Thermal shock waves under a Maxwell–Cattaneo model with temperature-dependent conductivity

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The propagation of thermal shock waves under the Maxwell–Cattaneo hyperbolic heat conduction theory is considered. In particular, the thermal conductivity is allowed to dependent on the temperature, which makes the model nonlinear. A solution for the evolution of a discontinuity under the linearized version of the governing equations is found via singular surface theory. Then, the Rankine–Hugoniot conditions are established for the nonlinear heat conduction model. The latter two results are used to derive a novel approximate solution to the problem of a Heaviside pulse propagating into a half space. Finally, this approximate analytical expression is compared to the numerically-computed solution of the governing equations, and its range of validity is established.

Key Words: *Shock waves, Hyperbolic heat conduction, Rankine–Hugoniot conditions*

1 Introduction and problem formulation

Consider the one-dimensional flow of heat in a homogeneous rigid solid, occupying the positive x-axis, in the absence of any external or internal heat sources. Then, from the conservation of internal energy, we know that the absolute temperature ϑ and the heat flux $\vec{q} = (q(x,t),0,0)$ are related by

$$C_0\vartheta_t + q_x = 0, \qquad (1)$$

where C_0 is the heat capacity of the solid at some reference temperature $\vartheta_{\mathrm{R}} (\geq 0)$. To close the system, a constitutive relation connecting q to ϑ is necessary. Here, we use the well-known Maxwell–Catteneo flux law of hyperbolic heat conduction (see, e.g., Refs[1,2,3] and those therein for details):

$$\lambda_0 q_t + q = -K\vartheta_x, \qquad (2)$$

where λ_0 is the thermal relaxation time at the reference temperature. Following Refs[1,2,3], we assume a weakly temperature-dependent thermal conductivity, namely

$$K(\vartheta) = K_0[1 + \beta(\vartheta - \vartheta_{\mathrm{R}})], \qquad (3)$$

where K_0 is the conductivity at the reference temperature, and $|\beta| << 1$. In Ref.[1], it was shown that shock waves do not exist for $\beta < 0$. Therefore, we restrict to the case $\beta > 0$ because we are interested in studying shock propagation.

Now, we introduce the dimensionless variables

$$x^\circ = x/L,\ t^\circ = t(\kappa_0/L^2),\ \theta = (\vartheta - \vartheta_{\mathrm{R}})/\vartheta_0, \qquad (4)$$

$$q^\circ = q(L/(K_0\vartheta_0)),$$

where L and $\vartheta_0 (> \vartheta_{\mathrm{R}})$ are the characteristic spatial and thermal scales, respectively, and $\kappa_0 = K_0/C_0$ is the thermal diffusivity at the reference temperature. Then, leaving the superscript circles understood, the system of governing equations, i.e., Eqs(1,2,3), can be written in matrix notation as

$$\begin{pmatrix}\theta\\ q\end{pmatrix}_t + \begin{pmatrix}0 & 1\\ c_0^2(1+\varepsilon\theta) & 0\end{pmatrix}\begin{pmatrix}\theta\\ q\end{pmatrix}_x = -\frac{1}{\tau_0}\begin{pmatrix}0\\ q\end{pmatrix}, \qquad (5)$$

where $\tau_0 = \kappa_0\lambda_0/L^2$ is the Cattaneo number (or dimensionless relaxation time), $c_0 = 1/\sqrt{\tau_0}$ is the speed of propagation of infinitesimal disturbances and $\varepsilon = \beta\vartheta_0$. Since $\vartheta_0 > 0$ and we assumed $\beta > 0$, it follows that $\varepsilon > 0$ also. Note that the system given in Eq.(5) is strictly hyperbolic since the eigenvalues of the coefficient matrix, $\mu_{1,2} = \pm c_0\sqrt{1+\varepsilon\theta}$, are real and distinct for all x and t by the positivity of ε and θ.

At this point we are ready to formulate the initial-boundary-value problem (IBVP) we wish to solve. We are interesting in studying the evolution of a pulse propagating into a half-space occupied by a rigid heat conductor at rest at the reference temperature $\vartheta = \vartheta_{\mathrm{R}}$ ($\Leftrightarrow \theta = 0$). To this end, we supplement Eq.(5) with the initial and boundary conditions:

$$\theta(0,t) = H(t),\ \theta(\infty,t) = 0,\ 0 < t < \infty, \qquad (6)$$

$$\theta(x,0) = 0,\ q(x,0) = 0,\ 0 < x < \infty,$$

where $H(t)$ is the Heaviside unit step function. The first line above is just a statement of the fact that at $t = 0^+$ we suddenly begin heating the conductor at the left endpoint $x = 0$, but the temperature remains at the initial value in the region far away from the boundary, for all time. The second line attests to the fact that the solid is initially in thermal equilibrium with zero temperature.

Finally, a short note is in order. Since we allow the thermal conductivity to be non-constant and temperature-dependent, then it follows that the thermal diffusivity, $\kappa = K/C_0$, must be such also. Thus, we could equivalently state that the model herein features a temperature-dependent diffusivity, as done in Ref.[1].

2 Linear theory

For small thermal disturbances, we may linearize the system given in Eq.(5) (i.e., set $\varepsilon = 0$) to obtain

$$\begin{pmatrix}\theta \\ q\end{pmatrix}_t + \begin{pmatrix}0 & 1 \\ c_0^2 & 0\end{pmatrix}\begin{pmatrix}\theta \\ q\end{pmatrix}_x = -\frac{1}{\tau_0}\begin{pmatrix}0 \\ q\end{pmatrix}. \tag{7}$$

It is easily seen that, upon eliminating q, Eq.(7) reduces to the damped wave equation (DWE): $\theta_{tt} + \frac{1}{\tau_0}\theta_t - c_0^2\theta_{xx} = 0$. The propagation of discontinuities under Eq.(7) can be studied analytically using *singular surface theory* (see, e.g., Refs[4,5]). The latter affords significant theoretical insight into the physics of shock (see, e.g., Refs[4,6]) and acceleration (see, e.g., Refs[1,2,7,8]) wave phenomena in heat conduction and related fields.

Now, let $\Sigma(t)$ be a planar wavefront across which θ suffers a jump discontinuity. Such a wavefront is termed a *singular surface*. Then, we denote by $[[F]](t) \equiv F^-(t) - F^+(t)$ the amplitude of the jump in some quantity $F(x,t)$, where

$$F^{\pm}(t) = \lim_{x \to \Sigma(t)^{\pm}} F(x,t) \tag{8}$$

are the limits from the regions ahead of and behind Σ, respectively. Now, assuming that $[[\theta]](t=0) \neq 0$ and $\Sigma(0) = 0$, it can be shown (see, e.g., Refs[4,5]) that

$$[[\theta]]_{\mathrm{DWE}}(t) = \exp\left(-\frac{t}{2\tau_0}\right), \tag{9}$$
$$\Sigma_{\mathrm{DWE}}(t) = c_0 t,$$

where we have used the fact that $[[\theta]](t=0) = 1$ from Eq.(6). Above, the "DWE" subscript refers to the fact that these expressions describe the evolution of a jump discontinuity under the damped wave equation (DWE), which (as discussed above) is equivalent to Eq.(7).

While Eq.(9) fully characterizes the propagation of a thermal shock under Eqs(7,6), it does not give the complete solution for the temperature $\theta(x,t)$. This can be obtained using integral transform methods (see, e.g., Refs[4,9]). For our purposes, however, it suffices to use the following *ad hoc* approximate solution:

$$\theta(x,t) \approx H\left(1 - \frac{x}{\Sigma(t)}\right)\left(1 - \frac{x\{1 - [[\theta]](t)\}}{\Sigma(t)}\right), \tag{10}$$

which was shown to be quite accurate for short time in Ref.[9]. Recall that, in this case, Σ and $[[\theta]]$ are given in Eq.(9).

3 Nonlinear theory

As discussed in Refs[10,4], one cannot hope to obtain, in general, exact analytical expression for both the amplitude $[[\theta]](t)$ and location $\Sigma(t)$ of a shock wave under a *quasilinear* governing equation such as Eq.(5). In the previous section, we succeeded in doing so because Eq.(7) is linear, though singular surface theory is also applicable to broader class of *semilinear* equations (see, e.g., Ref.[6]).

All is not lost, however. For quasilinear systems, it is possible to derive an equation governing the evolution of $\Sigma(t)$ using the Rankine–Hugoniot (compatibility) conditions (see, Ref.[10] for an in-depth discussion).

To this end, we note that Eq.(5) can be written as a (hyperbolic) system of *balance laws*:

$$\begin{pmatrix}\theta \\ q\end{pmatrix}_t + \begin{pmatrix}q \\ c_0^2\left(\theta + \frac{1}{2}\varepsilon\theta^2\right)\end{pmatrix}_x = -\frac{1}{\tau_0}\begin{pmatrix}0 \\ q\end{pmatrix}. \tag{11}$$

Then, following Ref.[10], the Rankine–Hugoniot conditions take the form

$$[[\theta]]\frac{\mathrm{d}\Sigma}{\mathrm{d}t} = [[q]], \tag{12}$$
$$[[q]]\frac{\mathrm{d}\Sigma}{\mathrm{d}t} = [[c_0^2\left(\theta + \tfrac{1}{2}\varepsilon\theta^2\right)]],$$

where $\mathrm{d}\Sigma/\mathrm{d}t$ is termed the *shock speed*.

From Eq.(6), we know that $\theta^+(t) = 0 \quad \forall t \geq 0$, thus $[[\theta]] \equiv \theta^-$. This allows us to use the identity for the jump of a product (see, e.g., Refs[1,4]) to rewrite the right-hand side of the second line in Eq.(12) as $c_0^2[[\theta]]\left(1 + \frac{1}{2}\varepsilon[[\theta]]\right)$. Then, by eliminating the unknown quantity $[[q]]$ from the second line in Eq.(12), we obtain an equation for the shock speed in terms of the jump amplitude:

$$\left(\mathrm{d}\Sigma/\mathrm{d}t\right)^2 - c_0^2\left(1 + \tfrac{1}{2}\varepsilon[[\theta]]\right) = 0 \quad ([[\theta]] \neq 0). \tag{13}$$

Clearly, the only positive real solution of the latter is

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}t} = c_0\sqrt{1 + \tfrac{1}{2}\varepsilon[[\theta]](t)}. \tag{14}$$

For definiteness, we picked the positive root so that the singular surface Σ is right-propagating, this is due to our earlier assumption that the heat conduction takes place along the positive x-axis. Also, note that if we let $\varepsilon \to 0$ in Eq.(14), we recover the result from the linear theory (compare the result of taking this limit to the second line in Eq.(9)).

4 An approximate analytic solution

One shortcoming of the nonlinear theory is that it does not reveal much about the propagation of the shock wave beyond the relationship between the shock's speed and amplitude given by Eq.(14). An additional equation is needed for the full analytic solution to be obtained, and (as noted in Ref.[1]) this is the topic of ongoing research. The next best thing one can do is find asymptotic and/or approximate analytic solutions (see, e.g., Ref.[11]).

In this vein, a new approach to making this problem tractable was proposed in Ref.[4]. There, the idea is to make the approximation $[[\theta]] \approx [[\theta]]_{\mathrm{DWE}}$, where $[[\theta]]_{\mathrm{DWE}}$ is given in Eq.(9). Then, one can integrate Eq.(14) and obtain an analytic expression for the wavefront location to supplement the postulated amplitude expression. The result is

$$\Sigma(t) \approx 4\tau_0 c_0\left(\sqrt{1+\tfrac{1}{2}\varepsilon} - \sqrt{1+\tfrac{1}{2}\varepsilon\exp(-t/2\tau_0)}\right) \quad (15)$$

$$+4\tau_0 c_0 \ln\left(\frac{1+\sqrt{1+\tfrac{1}{2}\varepsilon\exp(-t/2\tau_0)}}{1+\sqrt{1+\tfrac{1}{2}\varepsilon}}\right) + c_0 t.$$

Again, it is easy to check that as $\varepsilon \to 0$, $\Sigma(t) \to \Sigma_{\mathrm{DWE}}(t)$, showing that this is a consistent approximation.

The final step in constructing our approximate analytic solution to the IBVP given by Eqs(5,6) is to use the conclusion in Refs[4,6] that the *ad hoc* solution to the DWE, given above in Eq.(10), is in good agreement with the shape of the wave behind the wavefront even for the original nonlinear wave equation. Therefore, upon substituting the expression for Σ from Eq.(15) and the expression for $[[\theta]]$ from Eq.(9) into Eq.(10), we have our approximate analytic solution.

5 Comparisons with the numerical solution

In this section, we study the range of applicability of the *ad hoc* solution proposed in the previous section. To this end, we solve Eq.(5) subject to the initial and boundary condition given in Eq.(6) using a high-resolution shock-capturing (i.e., Godunov-type) numerical scheme, the details of which can be found in Refs[4,6]. This provides an accurate reference (exact) solution against which we can compare the linear and approximate nonlinear theories presented above.

Note that the model, at this point, has two free parameters, namely ε and τ_0. Therefore, for definiteness and simplicity, we henceforth take the dimensionless speed of propagation of infinitesimal disturbances to be unity, i.e., $c_0 = 1$ ($\Leftrightarrow \tau_0 = 1$). In the comparative studies below, we vary the "nonlinearity parameter" ε to determine the applicability of our approximate theory in the various regimes (i.e., small ε, large ε, etc.).

In each of Figs.1,2, we show the three solutions of the IBVP under consideration at two different instants of time. In each plot, the gray dash-dotted line corresponds to the *ad hoc* solution of Eqs(7,6) (i.e., the linearized problem) given by Eqs(10,9). The dashed black line corresponds to the *ad hoc* approximate analytical solution of Eqs(5,6) given by Eqs(10,15) with the expression for $[[\theta]]$ from Eq.(9). Finally, the thin solid black line represents the numerical solution of Eqs(5,6).

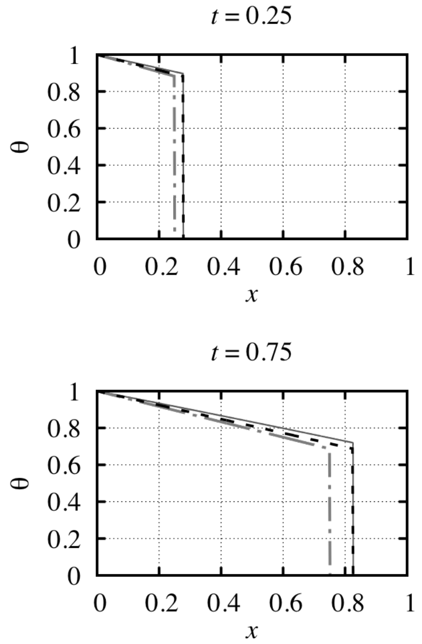

Figure 1: $\varepsilon = 0.5$.

Clearly, we obtain excellent qualitative agreement between the *ad hoc* approximate nonlinear solution and the numerically-computed exact solution, even for $\varepsilon, t = O(1)$. On the other hand, the solution to the linearized equations is only appropriate for $\varepsilon, t << 1$, i.e., for weak nonlinearity and short time, as one would expect.

6 Conclusions and outlook

In this paper, we presented a study of thermal shock waves under a nonlinear Maxwell–Cattaneo hyperbolic heat conduction model featuring a temperature-dependent conductivity. The solution to the linearized problem was discussed, and an approximation technique for the nonlinear problem was developed. In particular, we constructed an approximate analytic solution to the original (nonlinear) equations using the exact shock speed from the Rankine–Hugoniot conditions, the shock amplitude expression from the linear problem (i.e., the DWE) and a linear interpolation of the solution between the shock front and the domain boundary.

We found that our approximate analytic solution is in very good agreement with the actual (numerically-computed) solution to the IBVP. It appears, however, for longer times the amplitude of the shock waves is not captured as accurately as the wavefront location. One way to remedy this situation is to replace $[[\theta]](t)$

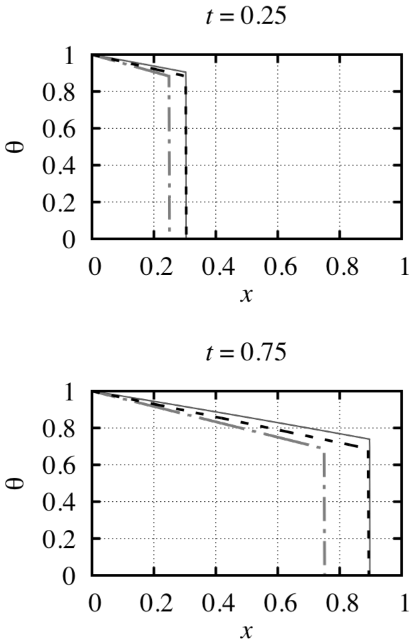

Figure 2: $\varepsilon = 1$.

with $[[\theta]](t+\phi)$ in Eq.(10), where ϕ is a phase shift. Unfortunately, the theory presented above does not give any insight into how to pick such a phase shift. Obviously, by comparing the approximate analytic and numerical solutions at various times, the parameter can be tuned by trial-and-error. More rigorous approaches to improving the match between the actual and *ad hoc* solutions are being investigated and will be the topic of future publications.

Additionally, the observations made in Section 5 regarding the quality of the approximate analytic solution agree with those made in Ref.[4], wherein the same approximation technique used above was developed for the nonlinear wave equation governing the transverse vibrations of a string in a resisting medium. This is not surprising as the latter equation can be re-interpreted as the governing equation of the problem studied in the present work (see Section 5.2 in Ref.[4]). Therefore, the approximation method from Ref.[4], which we presently applied to the study of thermal shocks, is both a general and extensible approach to bridging the theory gap between the exactly-solvable linearized equations and the (generally) theoretically-intractable fully nonlinear equations governing the dynamics of shock wave propagation.

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