The Crossing Number of a Graph

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1 Introduction

1.1 Motivation

Imagine you were the mayor of Zarankiewiczburg, a beautiful city on the Danube. You are faced with the following problem - you have to construct 6 bridges across the Danube that connect 6 suburbs on both sides. Being an idealist, you want to connect each suburb on one side of the river to all the ones on the other side of the river. You quickly realize this looks like an impossible task (unless you build some bridges higher vertically than others, but you do not have the engineering skills to do that). You convince yourself that the problem does not have a solution and you promptly give up.

In fact, the above problem does not have a solution, since the graph $K_{3,3}$ is not planar as we shall discuss later. The latter problem generalized to graphs is known is a problem of determining which graphs the planarity of the graph. In turn, the planarity problem is closely related to the crossing number problem, which this paper will focus on.

1.2 Definitions

We first introduce the following concepts and definitions in order define the crossing number problem.

Definition 1 *A drawing of a graph, $G = (V, E)$, is an embedding of G into $\mathbb{R}^2$. In other* *words, a mapping of all vertices, $v \in V$, to points in $\mathbb{R}^2$ and all edges, $e \in E$, to Jordan arcs terminating at the vertices the edge connects.*

A Jordan curve is homeomorphic to the unit circle, hence a Jordan arc is a piece of finite length of a Jordan curve. We would like to use Jordan arcs as edges because smoothness and continuity of the edges will thus be guaranteed. From this point on we shall denote a drawing of a graph G by $D(G)$.

Definition 2 *A drawing of a graph is "good" if and only if all edges intersect at most once.*

Note that the above definition requires that the intersections between two edges must occur at a vertex or at a *crossing* point, in other words all edges will not necessarily cross, but will intersect.

Definition 3 *Given a good drawing $D(G)$ of a graph $G = (V, E)$, the crossing number of the drawing denoted by $cr(D(G))$ is the number of single point crossings (non-vertex intersections) between edges that occurs in the drawing.*

A particular type of planar embedding of a graph will become important when defining the rectilinear crossing number of a graph, hence we shall define the rectilinear drawing of a graph now.

Definition 4 *A rectilinear drawing $\bar{D}(G)$ of a a G is an embedding of G into $\mathbb{R}^2$ such that every edge is mapped to a line in the plane connecting two vertices, which are mapped to points in the plane. Moreover, no three vertices can be colinear.*

Notice, in Definition 4 there is no need for a good drawing, since by requiring that no three vertices are colinear, then no edges can intersect continuously or more than once. The latter follows from elementary geometry since two lines in a plane either do not intersect at all, intersect at exactly one point or are indeed the same line and intersect at all points.

We can now define the crossing number of a graph (as opposed to the crossing number of a drawing of a graph).

Definition 5 *The crossing number of a graph* $G = (V, E)$ *is* $\nu(G) = \min\{cr(D(G))\}$.

Essentially the crossing number of a graph is the minimum crossing number of any of its drawings. Furthermore we can define the rectilinear crossing number of a graph, $\bar{\nu}(G)$, the exact same way, except we would have to replace D with $\bar{D}$.

Problem 1 *(Crossing Number Problem) Given a graph* $G = (V, E)$ *what is* $\nu(G)$*?*

1.3 Historical Background

Historically the problem is known as Turàn's Brickyard Problem, after P. Turàn who was the first to suggest it. During World War II Turàn's regiment was working a brick factory where every brick oven was connected to every storage hangar by rail. Rail carts were pushed on the rails from the ovens to the hangars. As Turàn explains in the introductory remarks of the first volume of the Journal of Graph Theory, the carts often derailed where different tracks crossed. Hence it occurred to him that it would have been a good idea if the builders of the brickyard had attempted to minimize the crossings of the rails. The latter is essentially the problem of the crossing number of a complete bipartite graph, which we shall discuss.

In 1954 Zarankiewicz provided what he believed was a proof for the counting number of the complete bipartite graph [11]. However, in 1969 R. K. Guy found an error in Zarankiewicz's proof [5]. Hence Zarankiewicz's calculation is not the exact crossing number for the complete bipartite graphs, but it is at least an upper bound to the exact value. and went on to conjecture the crossing number of the complete graphs. In [5], Guy initiated the search for the crossing number of the complete graphs offering a conjecture as to what the number might be. In 1971, Kleitman in a landmark paper, [6], proved that Zarankiewicz's conjecture predicts the correct values for all complete bipartite graphs $K_{5,m}$, $K_{6,m}$ for all positive m.

Today Guy's conjecture on the crossing number of the complete graphs and Zarankiewicz's conjecture for complete bipartite graphs have neither been proven nor disproven. However, in 1983 Garey and Johnson proved the crossing number problem to be NP-Complete and conjectured it is likely to be intractable [4]. Many improved upper and lower bounds have been found for the crossing numbers of the complete and complete bipartite graphs, but no exact solution. Furthermore the general crossing number problem has not been attempted by any mathematician due to the complexity of it.

2 Crossing Numbers

As one can imagine in general the problem would be very hard to approach, especially if G is a graph whose properties we don't know. Hence some simplifications must be made in order to approach the problem and attempt to solve it. A natural approach would be to consider graphs such as the complete graph K_n or the complete bipartite graph $K_{n,m}$ whose structure is relatively simple.

The values of the crossing number of K_n for $n \leq 10$ have been computed numerically (and proven) to be 0, 0, 0, 0, 1, 3, 9, 18, 36, 60 respectively [9]. For $n \geq 11$ an estimate can be obtained as we shall proceed to show, but no proof exists.

2.1 Crossing Number of Complete Graphs

Problem 2 *Given a complete bipartite graph $K_{n.m}$, what is $\nu(K_{n,m})$?*

Proposition 1 (Zarankiewicz's Conjecture) $\nu(K_{n,m}) = \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$

Zarankiewicz's reasoning behind the conjecture is as follows. Place $\lfloor \frac{n}{2} \rfloor$ of the vertices on the positive x axis and $\lceil \frac{n}{2} \rceil$ on the negative x axis; similarly place $\lfloor \frac{m}{2} \rfloor$ of the vertices on the positive and $\lceil \frac{m}{2} \rceil$ on the negative y axis. Connect the vertices on the x axis with those on the y axis by drawing nm straight line edges. Hence this is a drawing on $K_{n,m}$, and the number of crossings can be determined to be exactly as stated above [10].

Problem 3 *Given a complete graph K_n, what is $\nu(K_n)$?*

Proposition 2 (Guy's Conjecture) $\nu(K_n) = \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$

The construction of this conjecture is a lot more involved and subtle that Zarankiewicz's. To summarize it, the vertices of K_n are mapped to top and bottom discs of a cylinder isomorphic to the unit sphere. Then these vertices are connected to form two complete graphs isomorphic to $K_{n/2}$ on the top and bottom discs. By considering the shortest helical paths on the cylinder between the vertices on the top and bottom discs the value of for the crossing number in Proposition 2 can be obtained.

Theorem 1 *For the complete bipartite graphs $K_{5,m}$ and $K_{6,m}$, the crossing numbers are exactly as given by Zarankiewicz's conjecture – $\nu(K_{5,m}) = 4\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$ and $\nu(K_{6,m}) = 6\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$.*

This result is due to Kleitman. See [6] for a proof of the theorem. Essentially, Kleitman shows several smaller results that together imply Theorem 1.

2.2 Planar Graphs

The graphs with crossing number 0 have special properties and significance as we will explore in this section.

Definition 6 *A graph $G = (V, E)$ is called planar if and only if it has crossing number $\nu(G) = 0$.*

Some examples are the complete graphs K_n for $n < 5$ and the complete bipartite graphs $K_{n,m}$ for $n, m < 3$. Furthermore, the corresponding graphs of Platonic solids (also know as regular polytopes) are also planar.

Theorem 2 *(Kuratowski's Theorem) A graph $G = (V, E)$ is planar if and only if it contains no subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$.*

A proof can be found in Kuratowski's orginal paper on the subject [7]. Considering K_5 and $K_{3,3}$ are the two complete graphs with crossing number 1, if a subdivision of these is present in a general graph G, then intuitively G should not have a planar drawing; indeed that is the case.

Theorem 3 *Given a planar graph $G = (V, E)$, its rectilinear crossing number is equal to the crossing number, that is to say $\nu(G) = \bar{\nu}(G) = 0$.*

A proof can be found in Fáry's article [2]. Essentially, Theorem 3 states that any planar graph has a drawing where all edges are straight lines.

2.3 Computational Complexity

Since no method to find the crossing number of a graph exists, then a different approach would be to consider the computational problem in hopes to learn more about the original problem. In a very terse paper in 1983 Garey and Johnson proved that the problem of deciding if $\nu(G) \leq K$, for some integer K, is an NP-Complete problem [4].

The proof consists of several polynomial-time reductions of the problem. A bijection is set from the *Bipartite Crossing Number* problem to the *Optimal Linear Arrangement* problem and then another one from the *Crossing Number* problem to the *Bipartite Crossing Number* problem. Since the *Optimal Linear Arrangement* problem has been proven NP-Complete already, then the two problems bijected to it must be NP-Complete as well. For reference, the *Optimal Linear Arrangement* problem is defined as "Given a graph $G = (V, E)$ and an integer K is there a one-to-one function $f : V \rightarrow \{1, 2, \ldots, |V|\}$ such that $\sum_{\{u,v\} \in E} |f(u) - f(v)| \leq K$?"

3 Rectilinear Crossing Numbers

3.1 Relationship to the Crossing Number

Proposition 3 *Given a graph $G = (V, E)$ then $\nu(G) \leq \bar{\nu}(G)$.*

Proof: Let us consider a graph $G = (V, E)$ and it's minimal rectilinear crossing drawing $\bar{D}^*(G)$ with corresponding rectilinear crossing number $\bar{\nu}(G)$. Also, consider the isomorphic minimal crossing drawing of G, $D^*(G)$ with crossing number $\nu(G)$. Construct an isomorphism $\psi : \bar{D}^*(G) \rightarrow D^*(G)$. In the construction of the isomorphism, extra crossing cannot be induced in $D^*(G)$, because that will clearly not yield a minimal crossing drawing (the minimal crossing rectilinear drawing can be a minimal crossing drawing). Furthermore, the number of crossings in $D^*(G)$ would be same as those in $\bar{D}^*(G)$ if no crossings can be eliminated. Hence we have shown that $\nu(G) \leq \bar{\nu}(G)$.

$\square$

In particular, a family of graphs for which strict inequality holding in the above theorem are the complete graphs on n vertices for $n \geq 10$.

3.2 Rectilinear Crossing Number of the Complete Graphs

The rectilinear crossing numbers for K_n have been proven and computed to be 0, 0, 0, 0, 1, 3, 9, 19, 36, 62 for $n \leq 10$ [9]. The result for K_{10} was proven very recently in an impressive paper by Brodsky et al in [1]. In general an upper bound due to Jensen exists and states that $\bar{\nu}(K_n) \leq \frac{7}{432}n^4 + O(n^3)$. A better estimate, including a lower bound was found in 2003 by Finch, his result shows that

$$0.290 < \frac{61}{210} \leq \lim_{n\to\infty} \frac{\bar{\nu}(K_n)}{\binom{n}{4}} \leq \frac{5}{13} < 0.385. \quad (1)$$

For the complete bipartite graphs, Zarankiewicz's conjecture (Proposition 1) gives a good estimate of the rectilinear crossing number (because of the construction of the conjecture), however it hasn't been proven correct or incorrect for this problem either.

3.3 Computational Complexity

Unlike the general crossing number problem, the rectilinear crossing number problem has not been proven to be an NP-Complete problem.

4 A Modified Crossing Number Problem

4.1 Embedding Graphs in $\mathbb{R}$

Let us consider the following simplified crossing number problem.

Problem 4 *(Linear Embedding Problem) Given a graph $G = (V, E)$, inject all vertices, $v \in V$, to a subset of the integers and map all the edges, $e \in E$, to Jordan arcs in the plane with endpoints at these integer values on a line. What is the optimal placement of the vertices such that the number of crossings between edges is minimum?*

Instead of mapping the vertices to points in the plain, we will map them to point on the the real line. The edges will still be mapped to Jordan arcs in the plane, otherwise the problem will become very elementary. There are two versions of the problem; one if we allow the arcs to be drawn only above the number line, and two if allow the arcs to be above and below. Furthermore, let us refer to the linear embedding problem's crossing number as $\mu(G)$ (for the case when edges are allowed above and below the number line) and $\mu^+(G)$ (for the strictly above or below case) in order to distinguish it from the planar crossing number $\nu(G)$.

4.2 Crossing Numbers of Linearly Embedded of Graphs

4.2.1 General Remarks

Proposition 4 *For any give graph $G = (V, E)$, the following relationship between its crossing numbers must hold $\nu(G) \leq \bar{\nu}(G) \leq \mu(G) \leq \mu^+(G)$.*

Proof: In order to prove the above inequality, we need to show that three separate inequalities hold, i.e. $\nu(G) \leq \bar{\nu}(G)$ and $\nu(G) \leq \mu(G)$ and $\mu(G) \leq \mu^+(G)$. By Proposition 3 we know $\nu(G) \leq \bar{\nu}(G)$. Furthermore, it is cleat that $\mu(G) \leq \mu^+(G)$ because by allowing edges both above and below allows us to possibly decrease the number of crossings in the minimal crossing drawing; also it is clear no new crossings can be added. Consider a plane embedded graph, there are 2 degree of freedom for the placement of each vertex. However in the linear embedding there is 1 degree of freedom. Thus there cannot be less crossings in the linear embedding than the planar embedding. Hence we have shown the third and last inequality $\nu(G) \leq \mu(G)$.

□

4.2.2 Complete Graphs

By elementary inductive arguments, we can easily derive the following If we restrict ourselves to the case when the edges can only be drawn above the number line of vertices, then $\mu^+(K_{1,m}) = \mu^+(K_{n,1}) = 0$, and $\mu^+(K_{2,m}) = \sum_{i=1}^{m-1} i = \frac{(m-1)(m-2)}{2}$, which implies $\mu^+(K_{n,2}) = \frac{1}{2}(n-1)(n-2)$, are obvious. Similarly, in the case where we allow vertices on above and below the number line, $\mu(K_{1,m}) = \mu(K_{n,1}) = 0$, $\mu(K_{2,m}) = \mu(K_{n,2}) = 0$, and $\mu(K_{3,m}) = \sum_{i=1}^{m-1} i = \frac{(m-1)(m-2)}{2}$, which implies $\mu(K_{n,3}) = \frac{1}{2}(n-1)(n-2)$. However any further attempt at computing the crossing numbers of the linear embeddings becomes quite difficult, similarly to the plane embedding problem. No general formula is know for the linear embedding crossing numbers of the complete or complete bipartite graphs.

4.2.3 Outerplanar Graphs

In this section we shall explore a particular case of the problem of determining whether if a graph G is planar, i.e. $\nu(G) = 0$, then it is also "linear," i.e. $\mu(G) = 0$.

Consider the planar complete bipartite graph $K_{2,3}$. Clearly $K_{2,3}$ has crossing number $\nu(K_{2,3}) = 0$. However, it cannot be linearly embedded such that either $\mu^+(K_{2,3}) = 0$ or $\mu(K_{2,3}) = 0$. In fact we can use the formulas discussed in the previous section to show that $\mu(K_{2,3}) = 0$ and $\mu^+(K_{2,3}) = 1$. Hence not all planar graphs have linear embedding crossing numbers equal to zero. The following theorem shows that the outerplanar graphs (a subset of planar graphs) are indeed remain "linear."

Definition 7 *A planar graph $G = (V, E)$ is said to be outerplanar if there exists a drawing in the plane, $D(G)$, such that all vertices, $v \in V$, lie on the outer face of the graph and no edges, $e \in E$, cross.*

Without loss of generality we can assume the vertices lie on an n-gon, where $n = |V|$.

Theorem 4 *An outerplanar graph $G = (V, E)$ can be linearly embedded with all edges above the diagonal such that it has linear embedding crossing number $\mu^+(G) = 0$.*

Proof: We shall show an isomorphism exists that maps an outerplanar graph to a linear embedding with 0 crossings. Given an outerplanar graph $G = (V, E)$ we begin by constructing a mapping $\phi : G \to G'$, $G' = (V', E')$, that maps all vertices $v \in V$ to the vertices of an n-gon, $v' \in V'$, where $n = |V|$. By Definition 7, we know ϕ will preserve all edges and the planarity of G, hence our construction is an isomorphism. We continue by constructing another map $\psi : G' \to G''$, $G'' = (V'', E'')$, such that all $v' \in V'$ are mapped to integers, $v'' \in V''$, on a straight line with the property $v''_0 < v''_1 < ... < v''_{n-1}$, again $n = |V'| = |V|$. Moreover, ψ will assign all v''_i in the following manner, pick any $v'_0 \in V'$ (i.e. a vertex of the n-gon) and map it to $v''_0 \in V''$, then traverse G' either clockwise or counterclockwise and map the vertices $v'_i \in V'$ to those in $v''_i \in V''$ such that the property $v_i < v_j$ if $i < j$ is preserved. Since all vertices of G' lie on its outer face (i.e. they are indeed the vertices on the circumference of the polygon) then we know every vertex is uniquely mapped to a vertex of G''. All edge are preserved as well. Hence ψ is an isomorphism.

Now we must show that $\mu^+(G'') = 0$. The result is fairly intuitive. Essentially ψ "unwraps" the polygonal graph G' and "straightens" out the polygon's circumference into a line. By Definition 7 we know no edges intersect inside the polygon. Hence when we "straighten" the polygon out and homeomorph the line segments in it to Jordan arcs none of these arcs will intersect if we assure their maxima occur sufficiently low in the vertical direction.

Hence given an outerplanar graph G we can find a linear embedding $G'' = \psi \circ \phi(G)$ such that $\mu^+(G'') = 0$. By Proposition 4, $\mu(G'') \leq 0$ but since the crossing number cannot be negative, then it must indeed be identically zero.

□

4.2.4 Minimizing the Linear Embedding Crossing Number

Consider some graph G, we shall give an overview of a possible method which will embed the graph onto $\mathbb{R}$ and minimize the number of crossings.

First consider the case that the edges can only be draw above the number line. Then draw the graph (no particular rules are specified at this point). Find the first vertex, where an arc (edge) begins and intersects with another edge. Select the arc beginning (ending) at the vertex that intersect the most other edges. If switching the position of this vertex with any other one in the graph eliminates a crossing, switch it. Continue this process for all vertices. Several passes through the linear arrangement of vertices will likely be required. If in a pass through the vertices no vertices are switched then the minimal crossing drawing has been found. This procedure is essentially a parallel to the bubble sort in computer science.

The case where we allow edges to be drawn above and below the number line is a bit different, since we have the ability to switch vertices and also reflect them about the number line. First step now would be to attempt to distribute all the edges evenly above and bellow the number line, so that crossings are decreased. Hence we make passes through the vertices reflecting edges about the number line if it reduces the crossings no vertices are reflected in a pass. Then we can proceed by the method above and we will have a minimal crossing drawing for the embedded graph.

Not unexpectedly, it has actually been shown that the problem of determining whether $\mu(G) \leq K$, for some positive integer K and graph G, is an NP-Complete problem [8]. However the parallel problem of determining whether $\mu^{+}(G) \leq K$ has not been proven NP-Complete. Hence no optimal method for minimizing the crossings exists, but the above discussion has some merit, since it is $\mathcal{O}(n^2)$ while checking the crossing of every possible permutation of the vertex arrangement is $\mathcal{O}(n!)$.

5 Further Research and Related Problems

Recently a connection between Sylvester's four-point theorem and the rectilinear crossing number has been found [3]. Several generalizations of Sylvester's theorem have been solved, which could lead to some insight into the crossing number problems.

Further research on the linearly embedded graph is also desirable. The problem is somewhat more manageable than the planar embedding one. Attempting to deduce a closed formula for $\mu(K_n)$, $\mu^{+}(K_n)$, $\mu(K_{n,m})$, $\mu^{+}(K_{n,m})$ should be possible with more work on the problem.

Not surprisingly, the crossing number has applications to engineering. For example, the crossing number can be used to estimate the chip area required for Very Large Scale Integration (VLSI) circuits. Also, Leighton and others proved in the early 1980s that the chip area required for the realization of an electrical network is closely related to the crossing number of the underlying graph. Optimizing numerical algorithms used to determine the crossing number, even though the problem is NP-Complete, would be of great benefit to such applications of the problem.

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