Nonoscillatory Central Schemes on Unstructured Triangulations for Hyperbolic Systems of Conservation Laws

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The Big Picture

- Motivation:
  - Why unstructured triangulations when (Cartesian) tensor product grids are so simple and efficient to implement?
  - How should we design "genuinely multidimensional" schemes and limiters on unstructured grids?
  - Need for a simple & fast predictor algorithm for the time-dependent extension of L^1-minimization FEM (J.-L. Guermond & B. Popov).
- Background:
  - On October 30 there were 18,630 articles on MathSciNet with primary or secondary MSC matching 35L* (Partial differential equations of hyperbolic type).
  - High-resolution central schemes: Nessyahu & Tadmor (1992) $\rightarrow$ Jiang & Tadmor (1998) $\rightarrow$ Kurganov & Tadmor (2000) $\rightarrow$ Kurganov, Noelle, Petrova (2001) ...
  - Extension to unstructured grids: Arminjon *et al.* (1997).
  - Semi-discere central-upwind version: Kurganov & Petrova (2005).

A Picture is Worth a Thousand Words

Source: `http://www.fluidmech.net/gallery/ss_cars/sscars.htm`

Outline of the Talk

1. Intro to hyperbolic systems of conservation laws.
2. Crash course on Godunov-type central schemes.
   - Derivation of a fully-discrete second-order-accurate scheme.
     - 1D Example: the Nessyahu–Tadmor scheme.
     - 2D Example: our new scheme on unstructured triangulations.
   - The minimum-angle-plane reconstruction.
   - How to construct the dual (or staggered mesh) of an unstructured triangulation.
3. Numerical results.
   - Scalar equation with *convex* flux: inviscid Burgers.
   - System of equation with *convex* flux: Euler equations of gas dynamics.
   - Scalar equation with *nonconvex* flux.

Statement of the Problem and Notation (I)

- Consider the initial-boundary-value problem for a $(d+1)$D *hyperbolic* system of conservation laws:

$$\begin{cases} \vec{q}_t + \nabla_{\vec{x}} \cdot \vec{F}(\vec{q}) = \vec{s}(\vec{q}, \vec{x}, t), & (\vec{x}, t) \in \Omega \times (0, T], \\ \vec{q}(\vec{x}, t = 0) = \vec{q}_0(\vec{x}), & \vec{x} \in \Omega, \\ \vec{q}(\vec{x}, t) = \vec{q}_{\mathrm{BC}}(\vec{x}, t), & (\vec{x}, t) \in \partial\Omega \times (0, T]. \end{cases}$$

- $\Omega \subset \mathbb{R}^d$ is the interior of a polygonal domain and $\partial\Omega$ its boundary.
- $\vec{q} = (q_1, \ldots, q_d)^\top$ is the vector of conserved quantities, $\vec{F} = (\vec{f}_1, \ldots, \vec{f}_d)^\top$ is the tensor containing the corresponding fluxes and $\vec{s}(\vec{q}, \vec{x}, t)$ is a source term.

Statement of the Problem and Notation (II)

- Bad news: classical solutions break down in finite time, weak solutions are not unique. For $d > 1$ there is essentially no existence, uniqueness and regularity theory.
- Good news: integral formulation requires less regularity, idea of vanishing viscosity and the Kružkov entropy solution concept can be used to select a unique weak solution.
- We restrict to the case $d = 2$ and $\vec{s}(\vec{q}, \vec{x}, t) = 0$:

$$\begin{cases} \vec{q}_t + \vec{f}(\vec{q})_x + \vec{g}(\vec{q})_y = 0, & (x, y, t) \in \Omega \times (0, T], \\ \vec{q}(x, y, t = 0) = \vec{q}_0(x, y), & (x, y) \in \Omega, \\ \vec{q}(x, y, t) = \vec{q}_{\mathrm{BC}}(x, y, t), & (x, y, t) \in \partial\Omega \times (0, T]. \end{cases}$$

Integral Form and the Finite Volume Method

- Idea: integrate the differential form of the conservation law over $\mathcal{V} \times [t^n, t^{n+1}]$ and apply Gauss' Divergence Theorem to obtain

$$\int_{t^n}^{t^{n+1}} \left(\frac{d}{dt} \int_{\mathcal{V}} \vec{q}(\vec{x}, t)\, dA + \oint_{\partial\mathcal{V}} \vec{f}(\vec{q}(\vec{x}, t))\nu_x + \vec{g}(\vec{q}(\vec{x}, t))\nu_y\, ds \right) dt = 0.$$

Using the Fundamental Theorem of Calculus, we get that

$$\frac{1}{|\mathcal{V}|} \int_{\mathcal{V}} \vec{q}(\vec{x}, t^{n+1})\, dA = \frac{1}{|\mathcal{V}|} \int_{\mathcal{V}} \vec{q}(\vec{x}, t^n)\, dA - \frac{1}{|\mathcal{V}|} \int_{t^n}^{t^{n+1}} \oint_{\partial\mathcal{V}} \vec{f}(\vec{q}(\vec{x}, t))\nu_x + \vec{g}(\vec{q}(\vec{x}, t))\nu_y\, ds\, dt.$$

- Godunov's idea: Approximate the cell-averages of $\vec{q}$ by a piecewise-constant function, which we can evolve "exactly."
- How do we find a "good" approximation to the flux integrals?

Godunov-type Schemes: 1D Example (I)

- Consider the 1D conservation law $\vec{q}_t + \vec{f}(\vec{q})_x = 0$.
- Introduce a mesh size Δx, and consider the Steklov sliding average

$$\bar{\vec{q}}(x,t) := \frac{1}{\Delta x}\int_{I_x} \vec{q}(\xi,t)\,d\xi, \quad I_x = \left\{ \xi \;\middle|\; |\xi - x| \le \frac{\Delta x}{2} \right\}.$$

- Then, integrating the conservation law over I_x gives the semi-discrete form

$$\frac{d}{dt}\bar{\vec{q}}(x,t) + \frac{1}{\Delta x}\left[\vec{f}\Big(\vec{q}(x+\frac{\Delta x}{2},t)\Big) - \vec{f}\Big(\vec{q}(x-\frac{\Delta x}{2},t)\Big)\right] = 0.$$

Godunov-type Schemes: 1D Example (II)

- Then, integrating over the temporal interval $[t^n, t^{n+1}]$ gives the fully-discrete, conservative scheme

$$\bar{\vec{q}}(x, t^{n+1}) = \bar{\vec{q}}(x, t^n) - \frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} \vec{f}\Big(\vec{q}(x + \frac{\Delta x}{2}, t)\Big)dt - \int_{t^n}^{t^{n+1}} \vec{f}\Big(\vec{q}(x - \frac{\Delta x}{2}, t)\Big)dt\right].$$

- Let $x_i = (i - 1/2)\Delta x$ be the center of each computational cell.
- Then, there are two methods of sampling the F.V. formulation:
  - (Upwind) We sample at the cell-centers x_i at all time steps ($\equiv$ Godunov's idea).
  - (Central) We alternate between cell-interfaces $x_{i+1/2} \equiv i\Delta x$ and the cell-centers x_i.

Central Schemes in 1D (I)

- Let $w(x,t) \approx \vec{q}(x,t)$ and recall Godunov's idea.
- <u>Anzats:</u> At each discrete time t^n the approximate solution is a piecewise polynomial

$$w^n(x) := w(x,t^n) = \sum_{i=1}^{N} p_i^n(x)\chi_i(x), \quad \chi_i(x) := \mathbb{1}_{I_i},$$

  where $p_i(x)$ is an algebraic polynomial supported on the interval $I_i \equiv I_{x_i}$, which is centered around the cell-center x_i.
- Sampling the sliding average equation at the cell-interfaces $x_{i+1/2}$ we obtain

$$\bar{w}^{n+1}(x_{i+1/2}) = \bar{w}^n(x_{i+1/2}) - \frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} \vec{f}\Big(w(x_{i+1},t)\Big)dt - \int_{t^n}^{t^{n+1}} \vec{f}\Big(w(x_{i-1},t)\Big)dt\right].$$

Central Schemes in 1D (II)

- Given p_i^n, we can compute exactly

$$\bar{w}^n(x_{i+1/2}) = \frac{1}{\Delta x}\left[\int_{x_i}^{x_{i+1/2}} p_i^n(x)dx + \int_{x_{i+1/2}}^{x_{i+1}} p_{i+1}^n(x)dx\right].$$

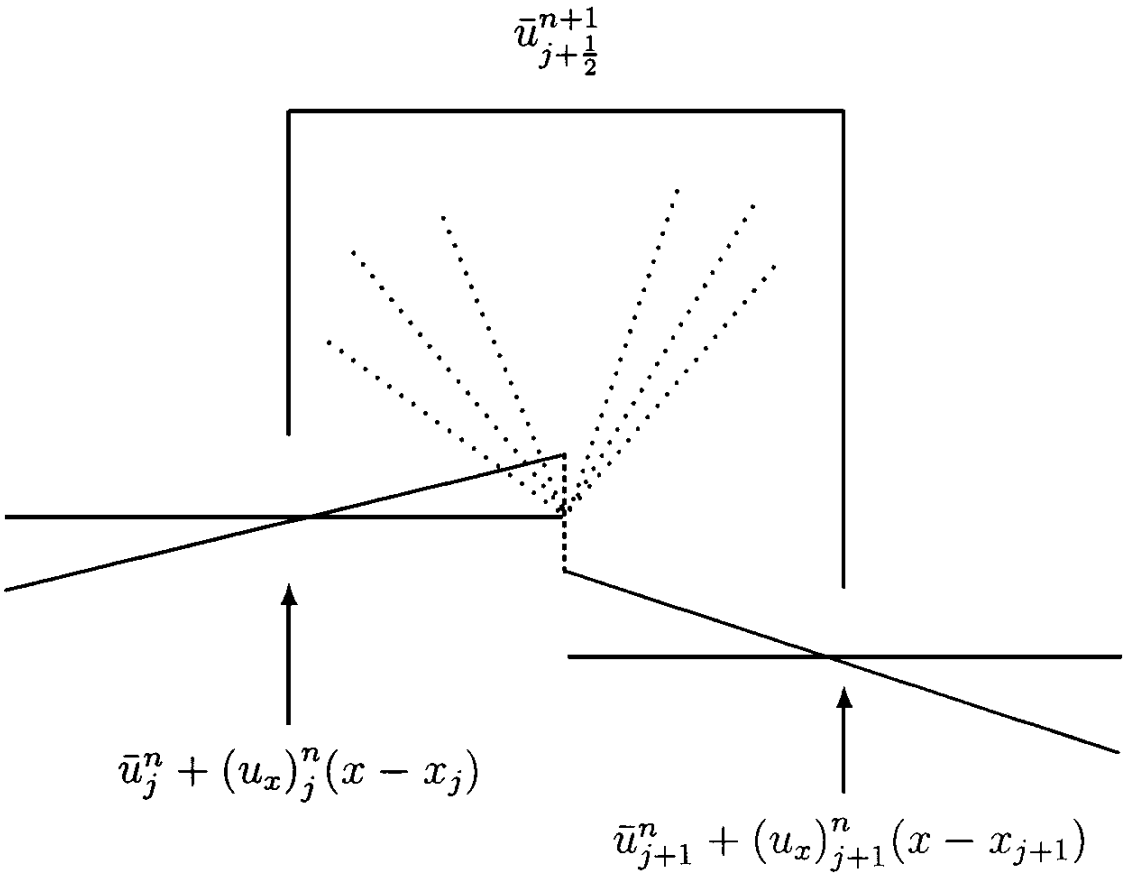
- And under the CFL condition $\max_i \rho\big(\frac{\partial \vec{f}}{\partial \vec{q}}(\bar{w}^n(x_i))\big)\Delta t < \frac{\Delta x}{2}$ the numerical fluxes can be computed with an appropriate quadrature rule, say the midpoint rule:

$$\frac{1}{\Delta x}\int_{t^n}^{t^{n+1}} \vec{f}\big(w(x_{i\pm1}, t)\big)dt \approx \frac{\Delta t}{\Delta x}\vec{f}\big(w(x_{i\pm1}, t^{n+1/2})\big),$$

where $t^{n+1/2} := \frac{1}{2}(t^n + t^{n+1})$.

The Second-Order Nessyahu–Tadmor Scheme (I)

1. Starting with the piece-wise constant approximation to the cell average $\bar{w}^n(\cdot)$, reconstruct a piece-wise linear function $w^n(\cdot)$.
2. Compute the staggered averages $\bar{w}^{n+1}_{i+1/2}$, which gives us $\bar{w}^{n+1}(\cdot)$.



The Second-Order Nessyahu–Tadmor Scheme (II)

One of the simplest (yet most robust!) second-order, nonoscillatory reconstructions is

$$p_i^n(x) = \bar{w}_i^n + \frac{w_i'}{\Delta x}(x - x_i), \quad w_i' = \mathrm{mm}(\bar{w}_{i+1}^n - \bar{w}_i^n, \bar{w}_i^n - \bar{w}_{i-1}^n),$$

where $\mathrm{mm}(a, b)$ is the minmod (i.e., minimum-modulus) function

$$\mathrm{mm}(a, b) = \frac{1}{2}(\mathrm{sgn}(a) + \mathrm{sgn}(b)) \min(|a|, |b|).$$

The Second-Order Nessyahu–Tadmor Scheme (III)

1. Predict the temporal midvalues for the quadrature rule by a Taylor series expansion:

$$\bar{w}_{i\pm1}^{n+1/2} := w(x_{i\pm1}, t^{n+1/2}) = \bar{w}_{i\pm1}^{n} - \frac{\Delta t}{2}\frac{\partial \vec{f}}{\partial \vec{q}}(\bar{w}_{i\pm1})\frac{w'_{i\pm1}}{\Delta x}.$$

2. Compute the new (staggered) averages

$$\bar{w}_{i+1/2}^{n+1} = \frac{1}{2}(\bar{w}_i^n + \bar{w}_{i+1}^n) + \frac{1}{8}(w'_i - w'_{i+1}) - \frac{\Delta t}{\Delta x}\left[\vec{f}\left(w_{i+1}^{n+1/2}\right) - \vec{f}\left(w_{i-1}^{n+1/2}\right)\right].$$

A Convergence Result for Central Schemes

For a scalar, one-dimensional, homogenous conservation law, the following is true:

Theorem (B. Popov and O. Trifonov)

Let f be strictly convex and assume that the CFL condition

$$\frac{\Delta t}{\Delta x} \max |f'| \leq \kappa$$

holds with κ sufficiently small. Then, the Nessyahu–Tadmor scheme converges to the unique (Kružkov) entropy solution of the conservation law.

$\longrightarrow$ Numerische Mathematik, Vol. 104, Number 4, 2006.

Preliminaries for the 2D Godunov-type Scheme

- $\mathcal{T} = \{\tau_i\}$ is a *conforming* triangulation of $\bar{\Omega}$.
- $\mathcal{S} = \{\sigma_k\}$ is the *staggered* grid — the "dual" of $\mathcal{T}$.
- Let $\bar{w}^n$ be a piecewise-constant function that approximates the cell averages of $\vec{q}$ on $\mathcal{T}$ at time t^n.

$$\bar{w}^n(x,y) = \sum_{i=1}^{N} \bar{w}_i^n \chi_i(x,y),$$

- Also, let $\bar{\mathfrak{w}}^n$ be the analogue of $\bar{w}^n$ on $\mathcal{S}$.
- The piecewise-linear reconstruction of $\bar{w}^n$ will have the form

$$w^n(x,y) = \sum_{i=1}^{N} p_i^n(x,y)\chi_i(x,y),$$
$$p_i^n(x,y) = \bar{w}_i^n + \nabla p_i^n \cdot (x - x_i^*, y - y_i^*)^\top.$$

Overview of the 2D Central Scheme

1. Perform a slope-limited piecewise-linear reconstruction on $\mathcal{T}$.

$$\bar{w}^n \longrightarrow w^n$$

2. Evolve the cell averages on the staggered grid $\mathcal{S}$ in time.

$$w^n \longrightarrow \bar{\mathfrak{w}}^n, \quad \bar{\mathfrak{w}}^n \longrightarrow \bar{\mathfrak{w}}^{n+1}$$

3. Project the solution from $\mathcal{S}$ back onto $\mathcal{T}$.

$$\bar{\mathfrak{w}}^{n+1} \longrightarrow \mathfrak{w}^{n+1}, \quad \mathfrak{w}^{n+1} \longrightarrow \bar{w}^{n+1}$$

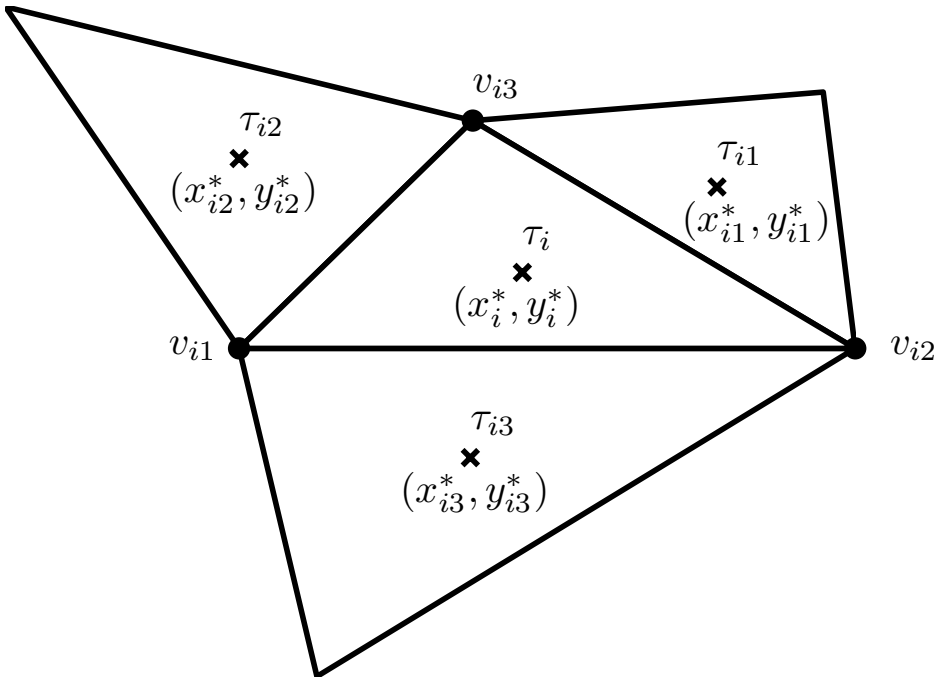
The good, the bad and the ugly:

- No need to solve a Riemann problem at each cell interface!
- Need to define $\mathcal{S}$ in a "reasonable" manner.
- Need to be able to perform a nonoscillatory reconstruction on $\mathcal{S}$.

The Minimum-Angle-Plane Reconstruction

The algorithm:

1. Given an element $\tau_i \in \mathcal{T}$ and its neighbors τ_{ij}, $1 \le j \le m$, find all $\binom{m+1}{3}$ possible planes.
2. Find the plane that makes the smallest angle with the horizontal, and use it to find a *limited* gradient.



Note that

- This "genuinely 2D" limiter behaves like minmod with a UNO flavor (i.e., $\approx$ Durlofsky–Engquist–Osher but $> 1^{\text{st}}$ order near extrema).
- No particular geometry and/or connectivity is assumed in the design of the limiter, and there are no *ad hoc* parameters.
- Limiter works exactly the same way on $\mathcal{S}$ as on $\mathcal{T}$.

Evolution Through the Staggered Averages

As before ...

1. Predict the temporal midvalues for the quadrature rule with via Taylor series expansion:

$$w(x,y,t^{n+1/2}) \approx w^n(x,y) - \frac{\Delta t}{2}\left(\frac{\partial \vec{f}}{\partial \vec{q}}\big(w^n(x,y)\big)\frac{\partial w^n}{\partial x} + \frac{\partial \vec{g}}{\partial \vec{q}}\big(w^n(x,y)\big)\frac{\partial w^n}{\partial y}\right).$$

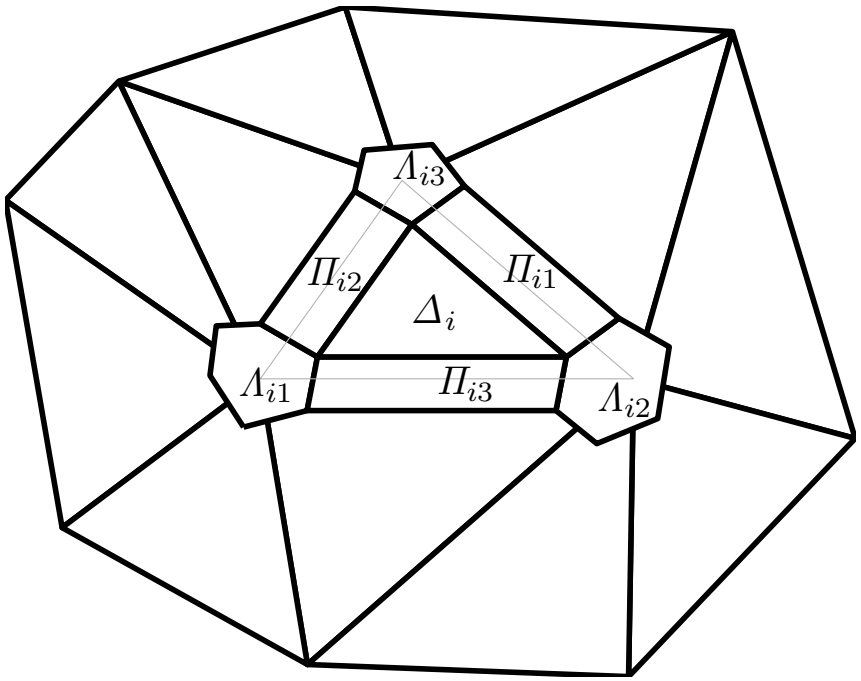
2. Compute the new (staggered) averages:

$$\bar{\mathfrak{w}}_k^{n+1} \approx \bar{\mathfrak{w}}_k^n - \frac{\Delta t}{|\sigma_k|}\oint_{\partial\sigma_k} \vec{f}\Big(\vec{w}(x,y,t^{n+1/2})\Big)\nu_x + \vec{g}\Big(\vec{w}(x,y,t^{n+1/2})\Big)\nu_y \, ds.$$

The Staggered Grid

The dual elements:

1. Triangles Δ_i.
2. Polygons Λ_{ij}.
3. Parallelograms Π_{ij}.



The usual CFL condition

$$\boxed{\Delta t < \frac{1}{3} \cdot \min_i |\tau_i| / S_{\max},}$$

$S_{\max}$ = fastest wave's speed, is good enough.

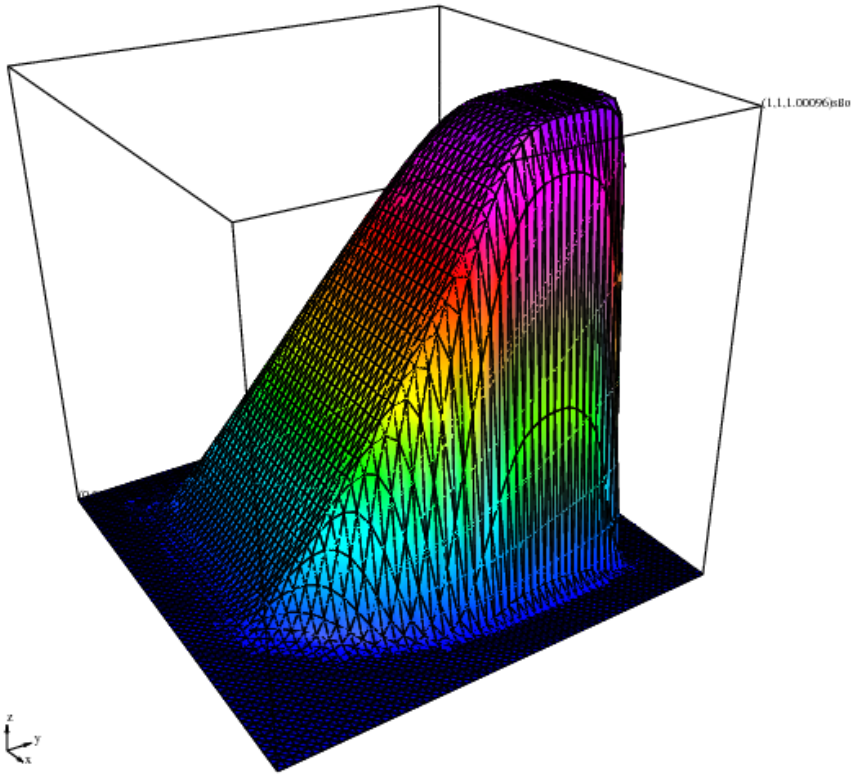
In particular:

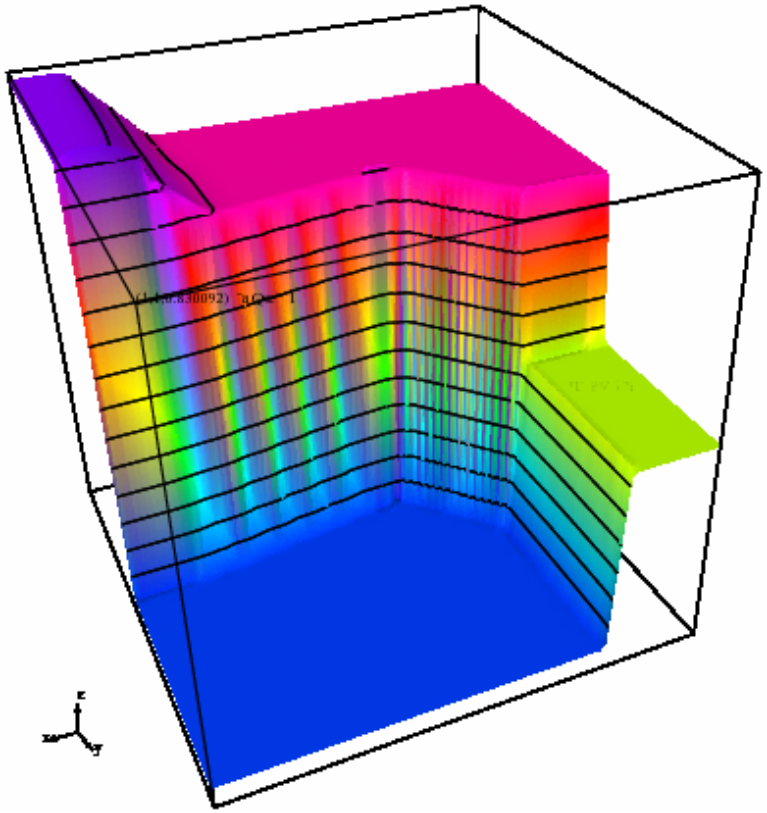
- If $|\Delta_i| = |\Pi_{ij}| = 0$, the staggered grid becomes the Voronoi diagram.
- If the maximum local speed of propagation is used, we get the analogue of Kurganov & Tadmor's modified central differencing for triangulations.

Numerical Results for a *Convex* Flux

- Riemann problems for the 2D inviscid Burgers equation

$$u_t + \left(\tfrac{1}{2}u^2\right)_x + \left(\tfrac{1}{2}u^2\right)_y = 0.$$

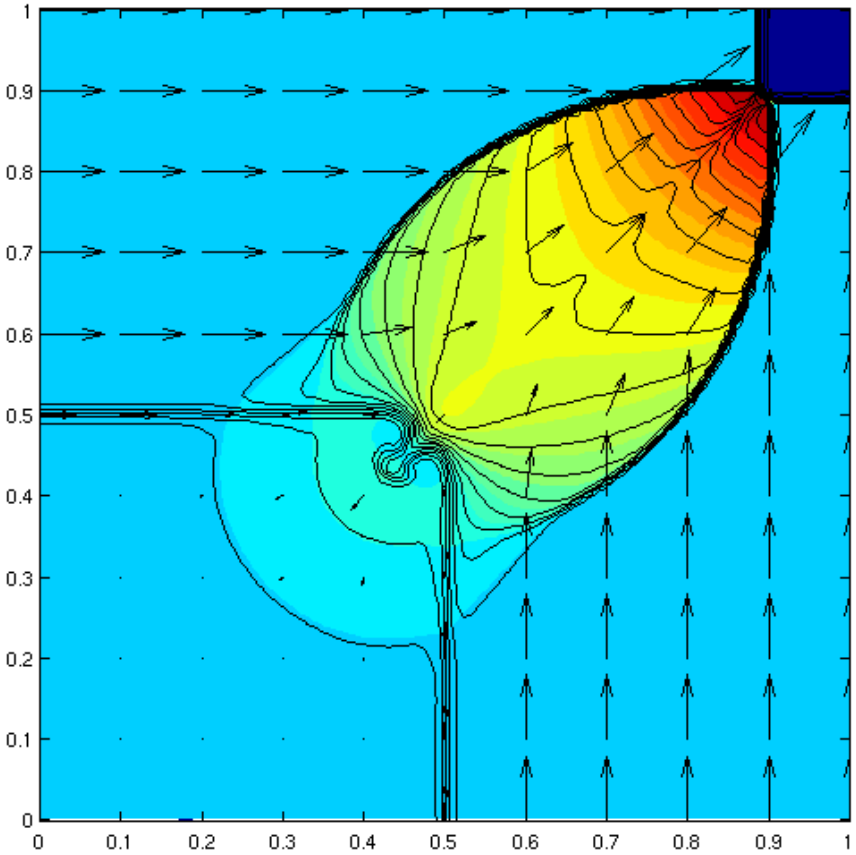


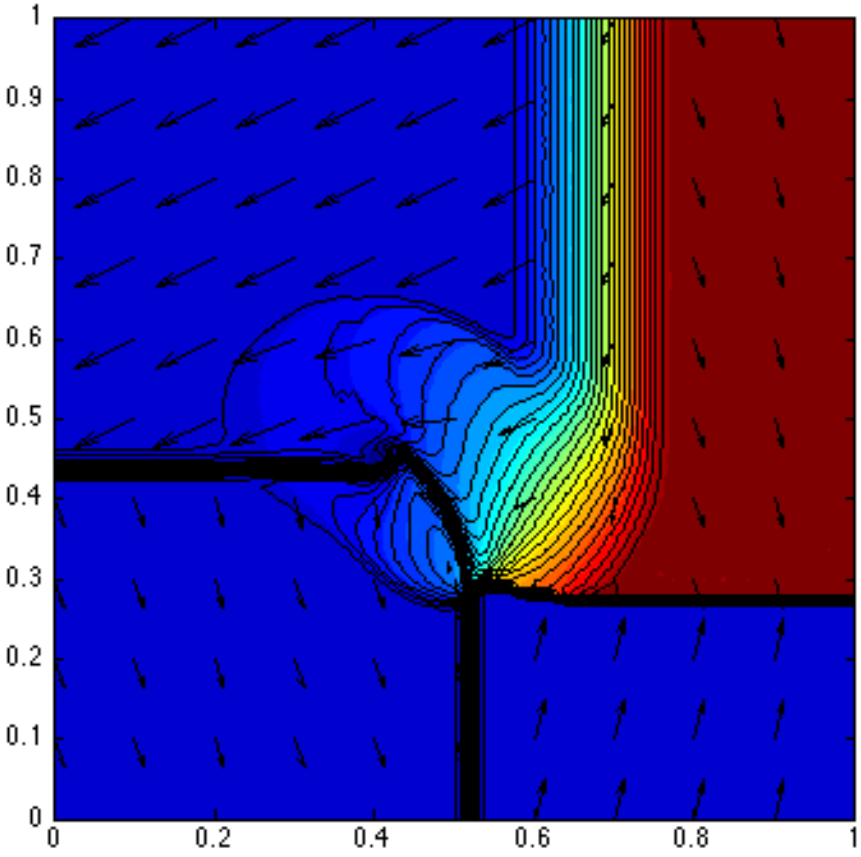


- (L) 6,272 elements and 19,041 dual elements.
- (R) 12,800 elements and 38,721 dual elements.

Riemann Problem for the Euler Equations

- $\vec{q} = (\rho, \rho u, \rho v, E)^\top$, $\vec{f}(\vec{q}) = \big(\rho u, \rho u^2 + p, \rho uv, u(E+p)\big)^\top$,
  $\vec{g}(\vec{q}) = \big(\rho v, \rho uv, \rho v^2 + p, v(E+p)\big)^\top$,
- For an ideal gas: $p = (\gamma - 1)\left[E - \frac{1}{2}\rho(u^2 + v^2)\right]$, $\gamma = 1.4$.



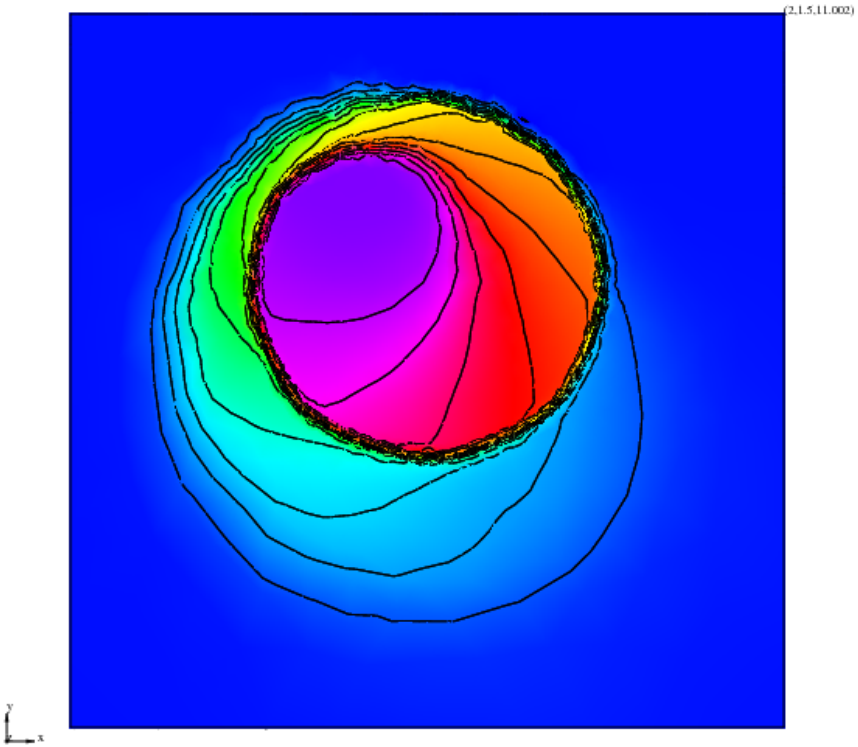


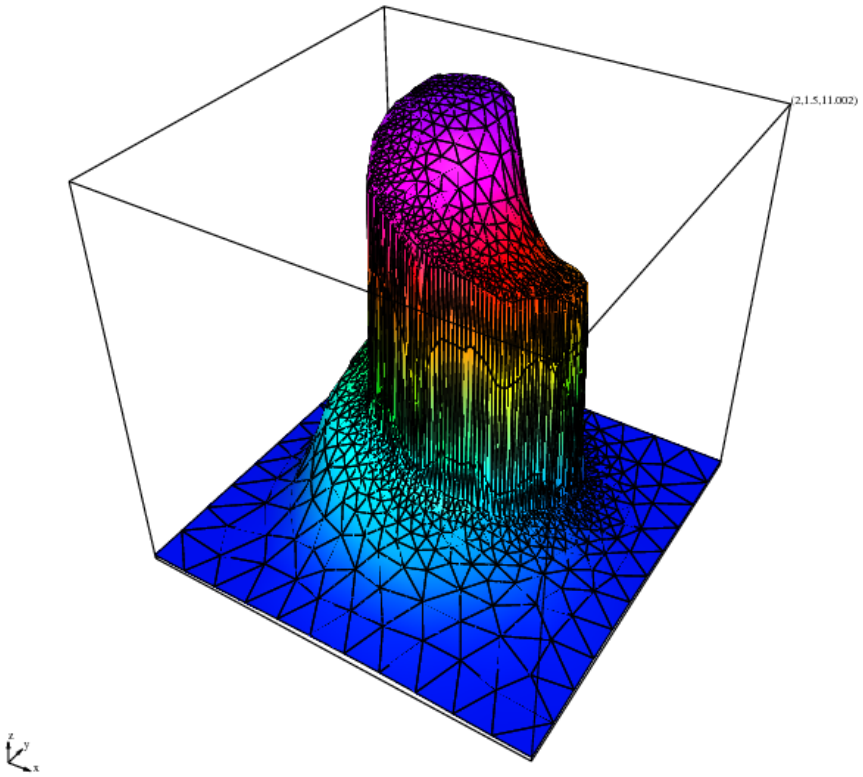
- Very fine mesh ($\min_i \operatorname{diam}(\tau_i) = \sqrt{2}/256$) with CFL $= 0.275$.

Numerical Results for a *Nonconvex* Flux (I)

- Riemann problem for the 2D scalar equation

$$u_t + (\sin u)_x + (\cos u)_y = 0.$$

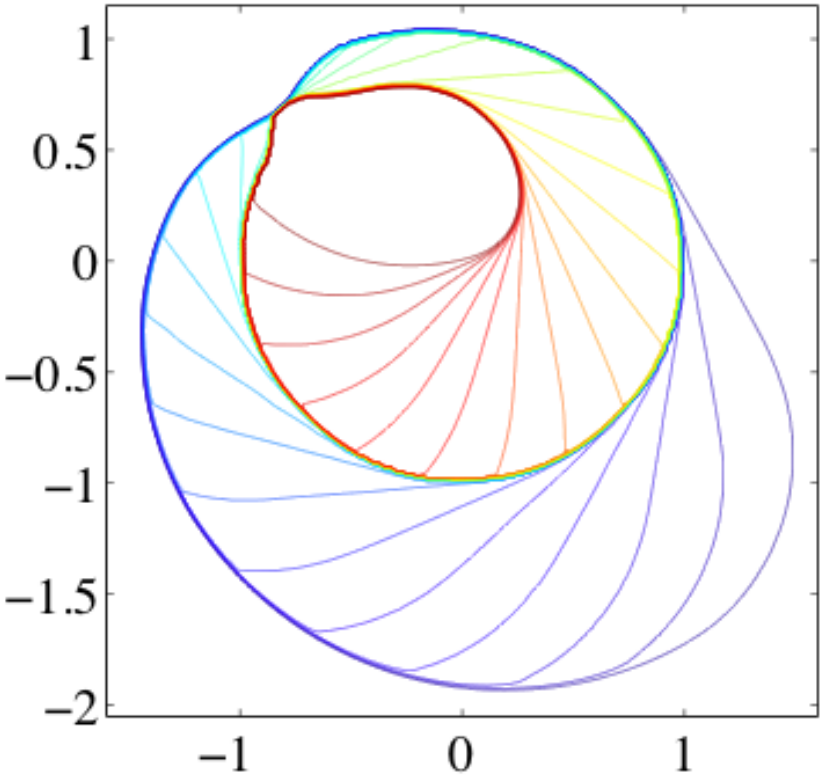


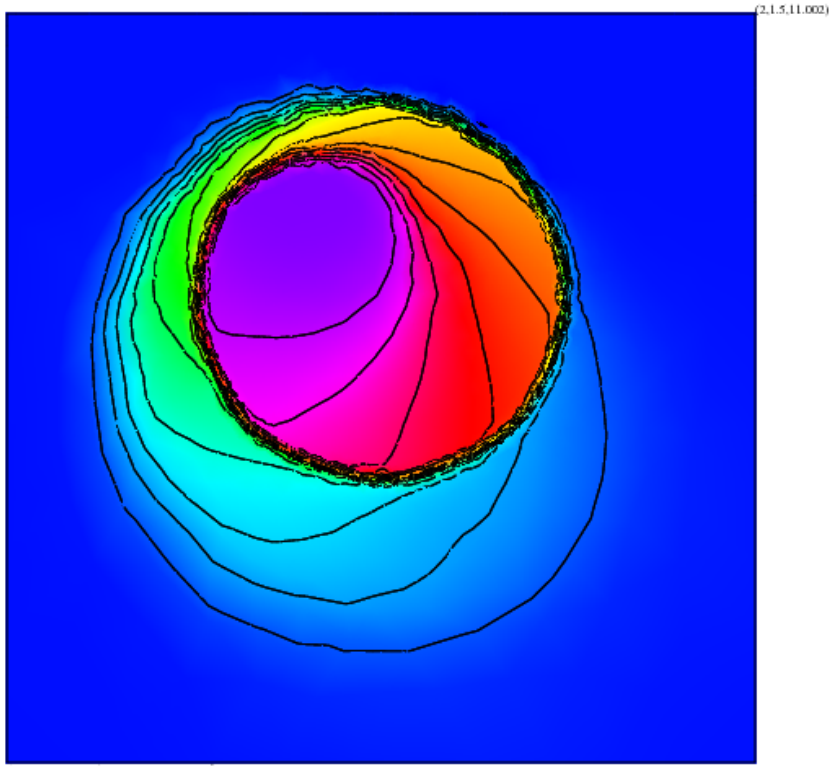


- Adapted mesh with (only) 3,264 elements and 9,837 dual elements, CFL $= \frac{1}{6}$.

Numerical Results for a *Nonconvex* Flux (II)

- Kurganov, Petrova & B. Popov reported that less compressive / higher order limiters (e.g., WENO5, MM2, SB) do *not* resolve the resulting composite wave correctly. The MAPR passes this test!





- (L) 90,000 element Cartesian tensor-product mesh.

Selected Bibliography

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