Chaotic granular mixing in quasi-two-dimensional tumblers: Streamline jumping, piecewise isometries and strange eigenmodes

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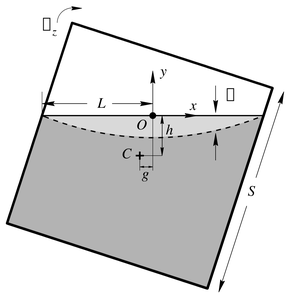
Dynamical Systems Framework for Tumbled Granular Flow

- Kinematic approach to mixing in a quasi-2D tumbled granular flow leads to the dynamical system

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \begin{cases} v_x(x(t), y(t), t), & y(t) > -\delta(x(t), t); \\ \omega_z[y(t) + h(t)] - \dot{g}(t), & \text{otherwise.} \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = \begin{cases} v_y(x(t), y(t), t), & y(t) > -\delta(x(t), t); \\ -\omega_z[x(t) + g(t)] - \dot{h}(t), & \text{otherwise.} \end{cases}$$

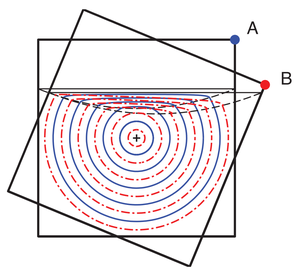


- From depth-averaged streamwise velocity $\bar{v}_x \neq \bar{v}_x(x)$ & conservation of mass (Khakhar *et al.*, *Chaos*, 1999) we get

$$v_x(x, y, t) = 2\bar{v}_x(t)\,[1 + y/\delta(x, t)]\,, \qquad v_y(x, y, t) = -\omega_z x\,[y/\delta(x, t)]^2\,,$$

$$\delta(x, t) = \delta_0(t)\left\{1 - [x(t)/L(t)]^2\right\}, \qquad \bar{v}_x(t) = \omega_z L(t)^2/[2\delta_0(t)].$$

- Streamline crossing criterion: to achieve chaotic mixing, superimposed streamlines at two different times should intersect.
- Changing length and depth of flowing layer ⇒ streamline crossing in a quasi-2D non-circular tumbler.
- Geometric similarity: $\epsilon := \delta_0(t)/L(t) = const.$ ⇒ flowing layer adjusts instantaneously to changes in container's orientation.
- $\delta_0 \sim 5$ to 12 particle diameters ⇒ $\epsilon \ll 1$.



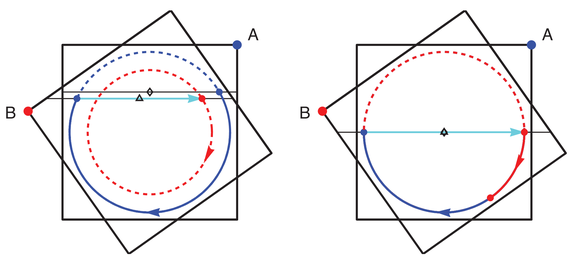
- If streamline crossing (⇒chaotic advection) were the only mixing mechanism, then the flow should become trivial as $\epsilon \to 0$. However, this is not true. **Why?**

Streamline Jumping: A New Mixing Mechanism

- $\lim_{\epsilon \to 0} \delta_0 = 0$ ⇒ flowing layer collapses onto the free surface becoming an interface.
- The average speed of a particle on the free surface becomes infinite:

$$\bar{V}_{\text{surf}} = \frac{1}{T_f}\int_0^{T_f}\frac{1}{2L}\int_{-L}^{+L}\sqrt{v_x^2 + v_y^2}\Big|_{y=0}\,\mathrm{d}x\,\mathrm{d}t = \frac{\bar{L}}{\epsilon} \to \infty \quad \text{as} \quad \epsilon \to 0.$$

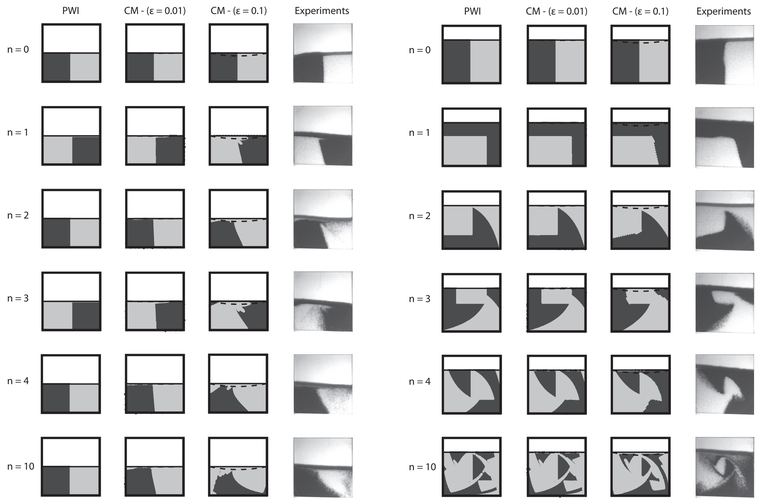
- For consistency $x_{\text{exit}} = -x_{\text{enter}}$ when crossing the flowing layer, $\tilde{x}_{\text{exit}} = -\tilde{x}_{\text{enter}} + 2g(t_{\text{enter}})$ w.r.t. C.



(a) $\phi = 0.73$ (b) $\phi = 0.50$

- Important point: $g(t_{\text{enter}}) \neq 0$ ⇒ streamline jumping because $\tilde{x}_{\text{exit}} \neq -\tilde{x}_{\text{enter}}$!

- Streamline jumping is a necessary and sufficient condition for chaotic mixing in a convex tumbler, rotated at a constant speed, with a vanishingly-thin flowing layer (Christov *et al.*, *PRE*, 2010).
- Can confirm this through comparisons of the model with experiments:



- As predicted, chaotic mixing is observed only for $\phi \neq 0.5$.
- Important point: $\epsilon = 0$ dynamics are the template (or "skeleton") of nearby ones.

Limiting Dynamics as a Piecewise Isometry

- The limiting ($\epsilon = 0$) dynamical system is a piecewise isometry (PWI) — a type of discontinuous dynamical system studied only recently (Goetz, *DCDS*, 1996).
- PWIs can exhibit the usual nonlinear dynamics (periodic points, fractal structure, global attractors, etc.) but there is no stretching and folding (Smale horseshoe) or chaos in the usual sense.
- The PWI is an affine transformation $\mathbf{\Omega}(t_1, t_2) : \mathcal{D}(t_1) \to \mathcal{R}(t_2)$ that can be written as

$$\mathbf{\Omega}(t_1, t_2) = \mathbf{T}(t_2) \circ \mathbf{R} \circ \mathbf{Q}(t_1, t_2) = \begin{pmatrix} -\cos[\omega_z(t_2 - t_1)] & -\sin[\omega_z(t_2 - t_1)] & 2g(t_2) \\ -\sin[\omega_z(t_2 - t_1)] & \cos[\omega_z(t_2 - t_1)] & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- Cutting and shuffling dynamics: $\mathbf{Q}$ "shuffles" by mapping each point in the flowing layer to a new location, $\mathbf{R} \circ \mathbf{T}$ "cuts" by reflecting & translating points along the flowing layer. This leads to spreading of material points in the continuum in the absence of stretching by shear strain.

Finite-Time Lyapunov Exponents and Manifold Structure

- Mixing and transport are affected by the underlying invariant structures of the flow, e.g., manifolds.
- So, consider the (largest) finite-time Lyapunov exponent (FTLE) field of the flow

$$\sigma(X, Y; t_0, \tau) = \frac{1}{|\tau|}\ln\sqrt{\Lambda_{\max}(\mathbf{C}(X, Y; t_0, \tau))},$$

- A Lagrangian coherent structure (LCS) is a ridge of σ; for a qualitatively analysis, can identify ridges with the darkest areas in the figure.
- Mass flux across an LCS is negligible (Shadden *et al.*, *Physica D*, 2005) ⇒ LCSs are finite-time analogues to the stable/unstable manifolds of the flows with arbitrary time-dependance.

(a) ε = 0.02 (b) ε = 0.01 (c) ε = 0.001 (d) ε = 0 Skeleton of the flow

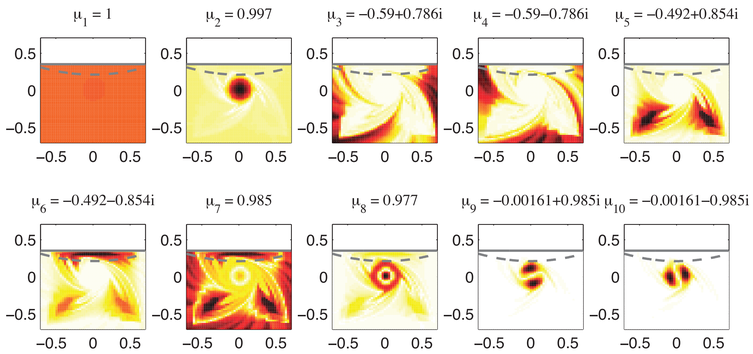
- Important point: manifold structure of the $\epsilon = 0$ system is the template or skeleton for all $\epsilon \ll 1$ ⇒ streamline jumping is the predominant mixing mechanism here (Christov *et al.*, *Chaos*, 2010).

Eigenmodes of a Tumbled Granular Flow

- The concentration C of a diffusive tracer being mixed is governed by an advection diffusion eq.

$$\partial_t C(\mathbf{x}, t) = \mathcal{L}C(\mathbf{x}, t), \qquad \mathcal{L} = -\mathbf{v}(\mathbf{x}, t)\cdot\nabla + Pe^{-1}\nabla^2.$$

- Asymptotic tracer distribution is determined by the dominant ("strange") eigenmodes (Pierrehumbert, *C.,S.&F.*, 1994). In time-periodic flows, introduce Floquet operator $\mathcal{F}$ s.t. $C(\mathbf{x}, t+nT) = \mathcal{F}^n[C(\mathbf{x}, t)]$.
- We can find the eigenmodes of the Floquet operator $\mathcal{F}$ by the simplified mapping method (Singh *et al.*, *Comput. Ch.E.*, 2009). Those with eigenvalues $|\mu_i| \approx 1$ persist for a very long time.
- For a monodisperse granular mixture (no segregation), strange eigenmodes form the asymptotic mixing pattern, but they give the segregation (unmixing) pattern for a bidisperse mixture.



$Pe = \infty$, $\epsilon = 0.2$, $\phi = 0.75$

Summary and Open Questions

✓ The $\epsilon = 0$ dynamics form the "skeleton," and mixing occurs via streamline jumping.

✓ The limit reveals new application of PWIs; LCSs show the manifold structure of this non-smooth flow.

✓ Strange eigenmodes exist in granular flows and determine the asymptotic advection pattern.

○ In 3D, what happens when multiple-axes rotation protocols are combined with streamline jumping?

○ Can we predict granular mixing *a priori* based on container geometry and PWIs? Can mixing protocols be designed to suppress the strange eigenmodes?

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