Pressure drop in a deformable channel distinguishes Newtonian from Boger fluids

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What do you picture when you hear "hydraulics"?

Figure: Industrial pipe network, https://www.steeljrv.com/

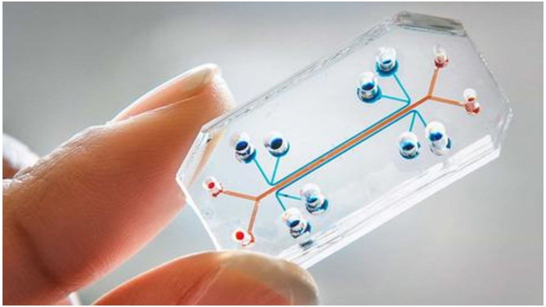


Figure: Microfluidic chip for chemical analysis, https://darwin-microfluidics.com/

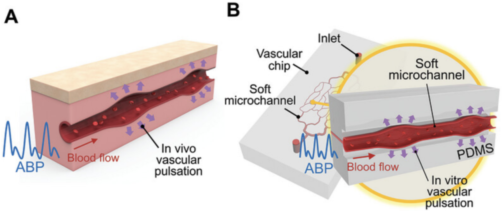
▷ Channel diameters ∼ meters, flow ∼ turbulent, hard materials ∼ steel ($E \sim 100$s GPa)

hard hydraulics

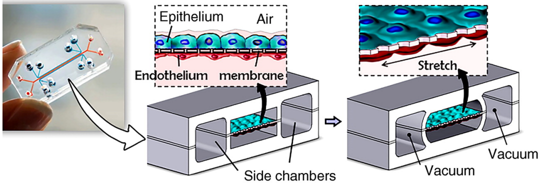
▷ Channel diameters ∼ 100s μm, flow ∼ laminar, soft materials ∼ PDMS (a gel, $E \sim 1$ MPa)

soft hydraulics

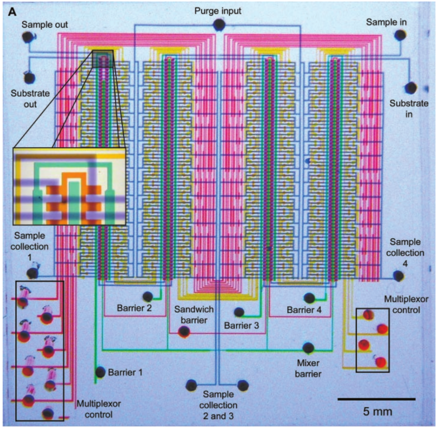
Soft hydraulics ☞ [blank]-on-a-chip



Artificial vasculature with pressure-responsive microfluidics
(Chen *et al.*, *Adv. Healthcare Mater.*, 2024)



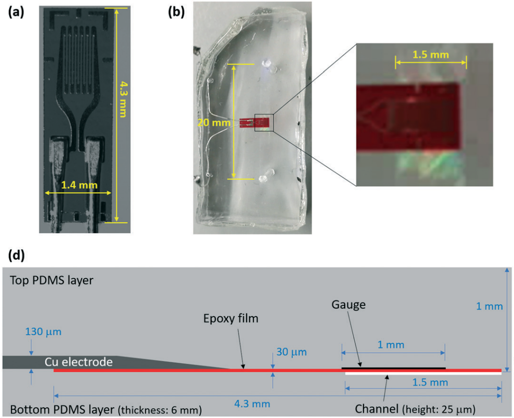
Organ-level lung functions on a chip (Huh *et al.*, *Science*, 2010)

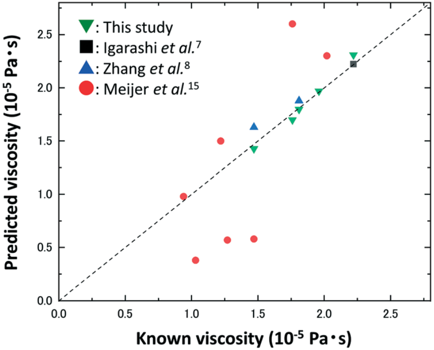


Microfluidic large-scale integration
(Thorsen, Maerkl & Quake, *Science*, 2002)

Soft hydraulics ☞ microrheometry

Exploiting channel compliance to measure the viscosity of gases:





(Shiba, ..., Weitz, *Lab Chip*, 2021; Shiba, Liu & Li, *Biosensors*, 2023)

- Used FEA to correlate viscosity to voltage from strain gauge sensor as $e_{\text{out}} \propto \mu^{2k}$, fitting k, expecting that $p \propto \mu^k$ (why? soft hydraulics!).

Soft hydraulics ☞ microrheometry — Open questions

BIOMICROFLUIDICS **10**, 043501 (2016)

CrossMark click for updates

Is microrheometry affected by channel deformation?

Francesco Del Giudice,[1,2,a)] Francesco Greco,[3] Paolo Antonio Netti,[1,2]
and Pier Luca Maffettone[1,2]

"Finally, we prove that the measure of the fluid relaxation time λ through particle migration in microfluidic devices might be unaffected by wall deformability if care is taken when selecting the flow conditions. Specifically, the dimensionless parameter $\Delta p/E$ must be small."

TODAY

How small? Is $\Delta p/E$ the only relevant parameter?
What are the basic laws/operating principles for **complex fluid flows** in compliant channels?

Soft hydraulics: Nearly unidirectional flows in compliant conduits

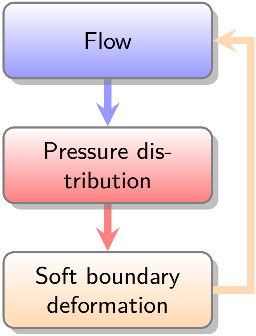


Figure: Fluid–elastic structure interactions (FSIs) in internal flows (following Fung, *Biomechanics*, 1997).

- Need ① deformation–pressure relation to determine change in cross-sectional area.
- Need ② the pressure grad.–flow rate relation:

$$\left(-\frac{dp}{dz}\right)\mathfrak{g}(\text{geometry}, \underbrace{p}_{\text{FSI}}, \underbrace{dp/dz, \ldots}_{\text{non-Newt.}}) = q.$$

This ODE for p updates classical result (Rubinow & Keller, *J. Theor. Biol.*, 1972).

- Them ③, the soft hydraulics problem is to find this ODE and solve it to get the relationship $\mathfrak{f}(\Delta p, q) = 0$.

① Deformation–pressure relation(s)

(a) Planar (2D), vertical deformation: $u_y(z) = \overbrace{\frac{b}{2G+\lambda}}^{\text{compliance [m/Pa]}} p(z)$

Shear modulus $G = E/2(1+\nu_s)$; Lamé coeff. $\lambda = E\nu_s/(1+\nu_s)(1-2\nu_s)$.

(*e.g.*, Winkler, 1867; Skotheim & Mahadevan, *PRL* 2004)

(b) Axisym. extrusion (2D), radial deformation: $u_r(z) = \frac{a_0}{2G} p(z)$

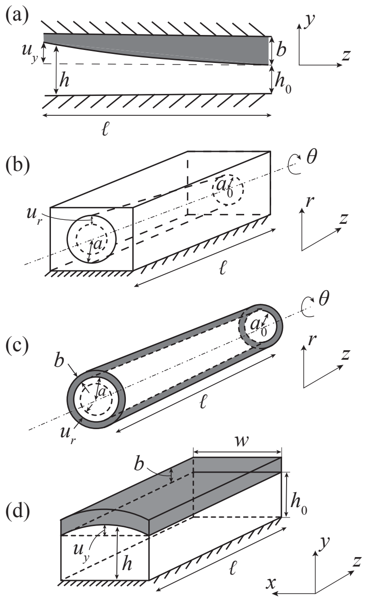
(*e.g.*, Wang *et al.*, *Mech. Res. Commun.*, 2022; Raj M *et al.*, *Biomicrofluidics*, 2018)

(c) Axisym. shell (2D), radial deformation: $u_r(z) = \frac{a_0^2}{b\bar{E}} p(z)$

$\bar{E} = E/(1-\nu_s^2)$ (*e.g.*, Anand & Christov, *ZAMM*, 2021; also Elbaz & Gat, *JFM*, 2014)

(d) 3D Cartesian, vertical deformation: $\boxed{u_y(x,z) = \frac{f(x)w}{\bar{E}} p(z)}$.

(Christov *et al.*, *JFM*, 2018; Shidhore & Christov, *JPCM*, 2018; Wang & Christov, *PRSA*, 2019)



From invited topical review: Christov, *J. Phys. Condens. Matter*, 2022.

② Pressure gradient–flow rate relation

- Long/slender geometry ⇒ lubrication approximation (≈ unidirectional flow, slow z variation):

$$0 \approx -\frac{\mathrm{d}p}{\mathrm{d}z} + \boldsymbol{\nabla}_\perp \cdot [\eta(\dot{\gamma})\boldsymbol{\nabla}_\perp v_z] + \boldsymbol{\nabla} \cdot (\boldsymbol{\tau}_{\text{polymeric}} \cdot \boldsymbol{e}_z), \qquad (*)$$

$\boldsymbol{\nabla}_\perp$ is in the cross-sectional (x, y) or (r, θ) coords.

⚠ Curving streamlines in nonuniform conduits cause viscoelastic stresses.

(Tichy, *J. Tribol.*, 1996; Zhang *et al.*, *JNNFM*, 2002; Saprykin *et al.*, *JFM*, 2007; Boyko & Stone, *JFM*, 2022, etc.)

- Flow rate is

$$q = \iint_{\text{deformed domain}(p)} v_z\,(\mathrm{d}p/\mathrm{d}z)\,\mathrm{d}A_\perp. \qquad (**)$$

- Using solution for v_z from $(*)$ and $u_{y \text{ or } r}$ from previous slide in $(**)$ gives ODE for $p(z)$:

$$-\frac{\mathrm{d}p}{\mathrm{d}z}\mathfrak{g}\left(\text{geometry}, p, \frac{\mathrm{d}p}{\mathrm{d}z}, \dots\right) = q,$$

③ Pressure drop of weakly viscoelastic flow in a 3D deformable channel

- Consider Boger fluid ($\eta(\dot{\gamma}) = \eta_s = const.$, $\eta_0 = \eta_s + \eta_p$), obeying Oldroyd-B model:

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \underbrace{2\eta_s \boldsymbol{E}}_{\text{solvent}} + \underbrace{\boldsymbol{\tau}}_{\text{polymeric}},$$

$$\boldsymbol{\tau} + \lambda[\boldsymbol{v}\cdot\nabla\boldsymbol{\tau} - (\nabla\boldsymbol{v})^{\top}\cdot\boldsymbol{\tau} - \boldsymbol{\tau}\cdot(\nabla\boldsymbol{v})] = 2\eta_p\boldsymbol{E}.$$

- Lubrication is not enough to make progress; assume weakly viscoelastic flow $De = \frac{\lambda q}{w h_0 \ell} \ll 1$; expand in De.
- Long story short (Boyko & Christov, *JNNFM*, 2023), leading contribution in De is

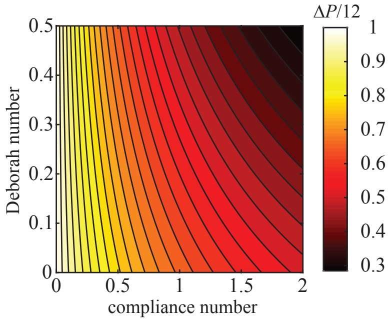
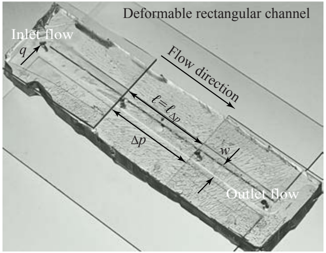
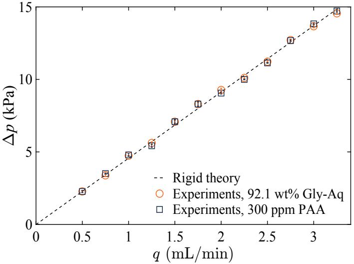


Figure: Wall compliance & viscoelasticity both decrease ΔP; $\eta_p/\eta_0 = 0.4$.

$$\Delta p \approx \underbrace{\frac{12\eta_0\ell}{w h_0^3}}_{R_h^{\text{rigid}}}\left[1 - \frac{3}{10}\underbrace{\left(\frac{w}{b}\right)^3\left(\frac{\ell}{h_0}\right)\left(\frac{\eta_0 q}{\bar{E} h_0^3}\right)}_{\text{wall compliance, }\alpha}\left(1 + 4\underbrace{\frac{\eta_p}{\eta_0}\frac{\lambda q}{w h_0 \ell}}_{\text{viscoelastic, }\tilde{\beta}De} + \cdots\right) + \cdots\right]q.$$

Viscoelastic soft hydraulics: quantitative theory–expt. agreement





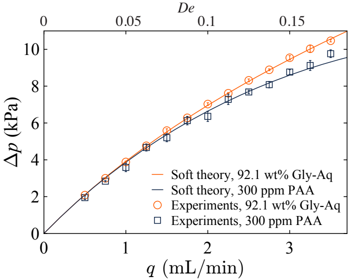


Figure: viscoelastic (Boger fluid, □) vs. Newtonian (glycerin, ○). (Chun, Christov, Feng, *Phys. Rev. Appl.*, 2025)

- Oldroyd-B model, 3D rectangular channel, $De = \lambda q/(w h_0 \ell) \ll 1$ (Boyko & Christov, *JNNFM*, 2023):

$$\Delta p \approx \underbrace{\frac{12\eta_0 \ell}{w h_0^3}}_{R_h^{\text{rigid}}} \left[1 - \underbrace{\frac{3}{10}\left(\frac{w}{b}\right)^3 \left(\frac{\ell}{h_0}\right)\left(\frac{\eta_0 q}{\bar{E} h_0^3}\right)}_{\text{wall compliance, } \alpha} \left(1 + 4 \underbrace{\frac{\eta_p}{\eta_0}\frac{\lambda q}{w h_0 \ell}}_{\text{viscoelastic, } \tilde{\beta} De} + \cdots \right) + \cdots \right] q.$$

- Here, $0.09 \le \alpha \le 0.56$ and $0.02 \le De \le 0.17$ based on varying inlet q.

Where did we come from? Where are we going?

- Got a handle on the soft hydraulic resistance $R_h(\Delta p)$. ✓
  - Including dependence on geometry (w/b, h_0/ℓ), channel wall softness (compliance #, α), viscoelasticity (Deborah #, De), shear thinning (Carreau #, Cu), . . .
- New operating principles for *soft* microrheometry? Infer λ from Δp?
  - Caveat: $\dot{\gamma}$ varies *slowly* with streamwise coordinate z.
- Stay tuned for oscillatory flows (Womersley #, Wo) two-way-coupled to deformation (elastoviscous #, γ).
  - Is there a soft hydraulic capacitance $C_h(\Delta p)$ [such that $q \approx C_h \frac{\mathrm{d}p}{\mathrm{d}t}$]?
  - Or inductance $L_h(\Delta p)$ [such that $p \approx L_h \frac{\mathrm{d}q}{\mathrm{d}t}$]?

Thank you for your attention!

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Rheological characterization of the working fluids

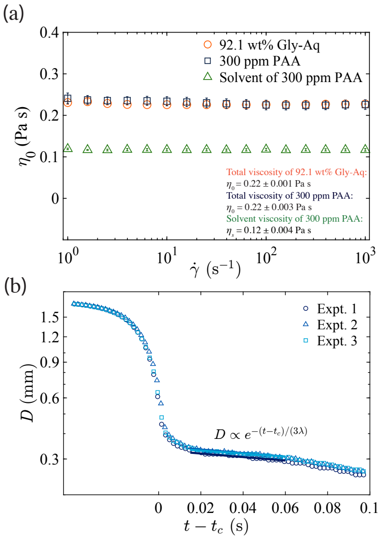
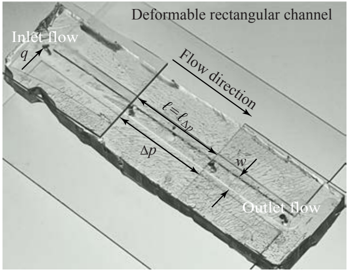


Table: Rheological properties and composition of the test liquids.

Name	Composition (% w/w)	$\eta_0 = \eta_s + \eta_p$ (Pa s)	η_s (Pa s)	λ (ms)
300 ppm PAA	PAA-0.03 Gly-89 DI water-10 NaCl-1	0.22 $\pm$ 0.003	0.12 $\pm$ 0.004	97.6 $\pm$ 0.5
Gly-Aq	Gly-92.1 DI water-7.9	0.22 $\pm$ 0.001	–	–

- Using (i) stress-controlled rheometer (DHR-3, TA Instruments) and (ii) custom-built dripping rheometry experiments.

Shear-thinning soft hydraulics: Quantitative theory–expt. agreement



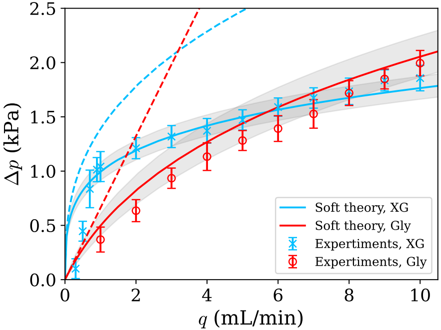


Figure: Shear-thinning (xanthan gum, ×) vs. Newtonian (glycerin, ○). (Chun, Boyko, Christov & Feng, *Phys. Rev. Fluids*, 2024)

Power-law model, 3D rectangular channel: $$-\frac{dp}{dz}\overbrace{\left\{1+\sum_{k=1}^{\infty} c_{k,n}\left[\frac{1}{384\tilde{b}}\frac{p(z)}{Bh_0/w^4}\right]^k {}_2F_1\left(\tfrac{1}{2},-k;\tfrac{3}{2}+k;\tilde{b}\right)\right\}^n}^{\mathfrak{G}(\text{geometry},p,\ldots)} = \frac{\Delta p_{\text{rigid}}}{\ell} = 2(4+2/n)^n\frac{\eta_0}{\lambda_r h_0}Cu^n$$

(Anand, Joshua David John Rathinaraj & Christov, *JNNFM*, 2019).