On a difficulty in eigenfunction expansion solutions for the start-up of fluid flow

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General Fluid Dynamics: Theory (L3.00010)

67th Annual Meeting of the APS DFD

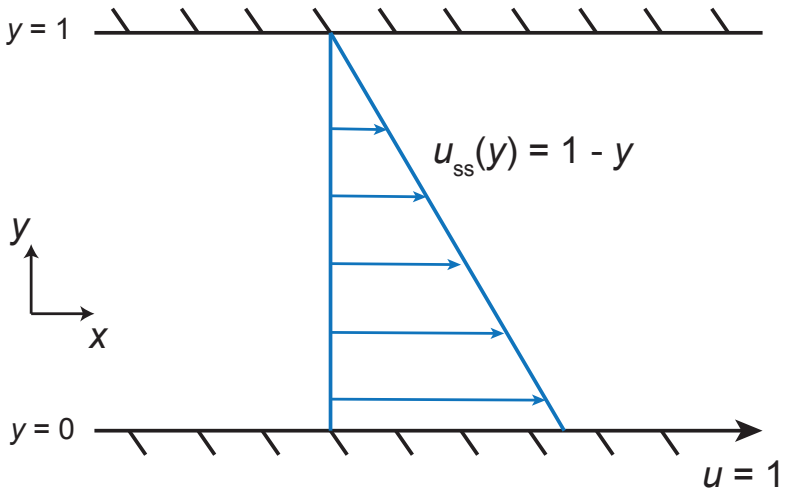
Boston, Massachusetts

November 23, 2015

Introduction: Start-up of fluid flow

- In 1851, G.G. Stokes considered the question of how momentum is transported in a fluid due to the impulsive motion of a boundary.
- For the unidirectional $\boldsymbol{u}(\boldsymbol{x},t) = \big(u(y,t),0,0\big)$ so $\boldsymbol{u}\cdot\nabla\boldsymbol{u} \equiv \boldsymbol{0}$ and $\nabla\cdot\boldsymbol{u} \equiv 0$.
- The resulting BVP, known as Stokes' 1st problem is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2},$$
$$u(y,0) = 0, \qquad 0 < y < 1,$$
$$u(0,t) = 1, \qquad t > 0,$$
$$u(1,t) = 0, \qquad t > 0.$$



- N.B.: u, y and t were rescaled to make BVP dimensionless.
- Solution can be found by standard techniques:
  $u(y,t) = (1-y) - \sum_{n=1}^{\infty} \mathrm{e}^{-n^2\pi^2 t}\,\frac{2}{n\pi}\sin(n\pi y),\ t > 0.$

Introduction: Generalization to non-Newtonian fluids

- Example non-Newtonian (viscoelastic) rheology

$$\sigma + \tau \frac{\partial \sigma}{\partial t} = \dot{\gamma} + \alpha \frac{\partial \dot{\gamma}}{\partial t}, \qquad \dot{\gamma} \equiv \frac{\partial u}{\partial y}.$$

- Eliminating σ using conservation of momentum, $\partial u/\partial t = \partial \sigma/\partial y$, Stokes' 1st problem becomes

$$\left(1 + \tau \frac{\partial}{\partial t}\right) \frac{\partial u}{\partial t} = \frac{\partial}{\partial y}\left(1 + \alpha \frac{\partial}{\partial t}\right) \frac{\partial u}{\partial y}, \qquad (y,t) \in (0,1) \times (0,\infty),$$

$$\begin{aligned} u(y,0) = \frac{\partial u}{\partial t}(y,0) &= 0, & 0 < y < 1, \\ u(0,t) &= 1, & t > 0, \\ u(1,t) &= 0, & t > 0. \end{aligned}$$

- What is the transient solution to this *third*-order BVP?

Textbook approach using eigenfunction expansions

(Carslaw and Jaeger, 1959; Batchelor, 1967; Leal, 2007; Bruus, 2008)

1. Obtain steady-state solution (elliptic problem): $u_{\text{ss}}(y) = 1 - y$.
2. Make BVP homogeneous via $u(y,t) = v(y,t) + u_{\text{ss}}(y)$, now $v(0,t) = v(1,t) = 0$ and $v(y,0) = -u_{\text{ss}}(y)$, $\frac{\partial v}{\partial t}(y,0) = 0$.
3. Solve homogeneous BVP via eigenfunction expansion, $v(y,t) = \sum_n a_n(t)\psi_n(y)$, where

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}\psi_n(y) = -\lambda_n \psi_n(y), \qquad \psi_n(0) = \psi_n(1) = 0.$$

4. Solve ODE for $a_n(t)$ using favorite method.

Textbook approach vs. Laplace transform & numerics

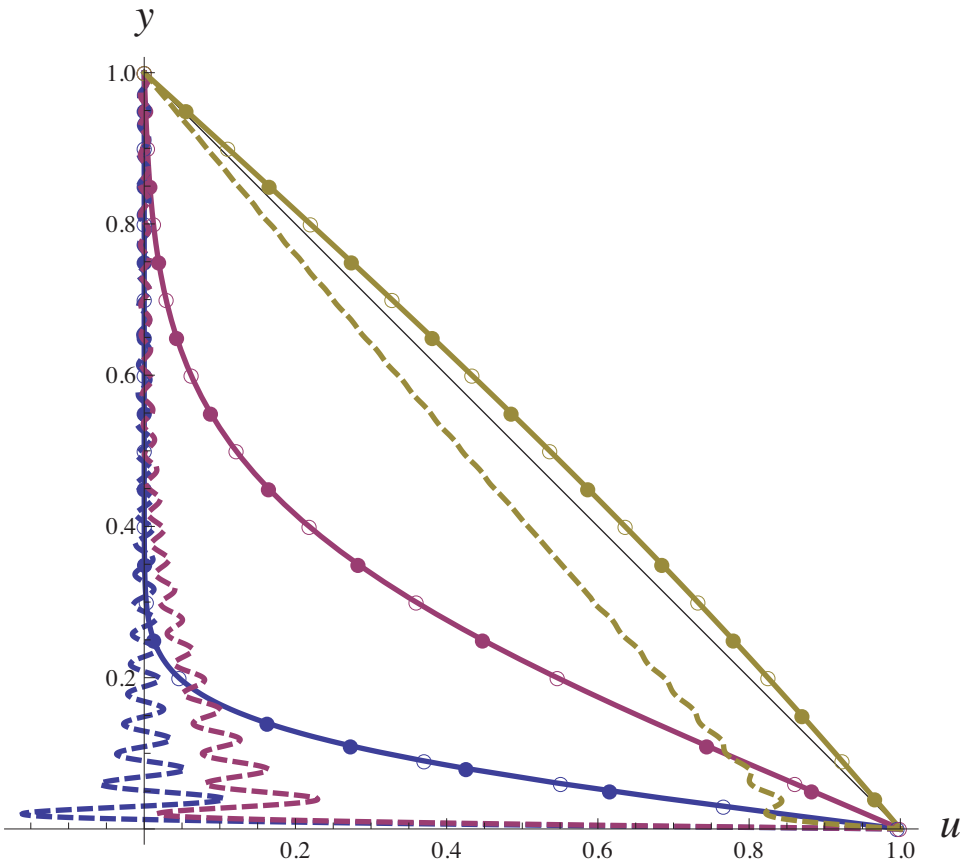


Figure: Dashed: textbook approach; Solid: Laplace transform in t; circles: numerical solutions.

Many such unphysical and incorrect solutions can be found in the literature:
Rajagopal (1982), Bandelli & Rajagopal (1995), Truesdell & Rajagopal (2000), Hayat *et al.* (2001, 2004), ...

A resolution to the apparent difficulty

1. Obtain steady-state solution: $u_{ss}(y) = 1 - y$.
2. Make BVP homogeneous via $u(y,t) = v(y,t) + H(t)u_{ss}(y)$, now $v(0,t) = v(1,t) = 0$ and $v(y,0) = 0$, $\frac{\partial v}{\partial t}(y,0) = 0$.
3. BVP is homogeneous but now includes forcing on the RHS:

$$\left(1 + \tau\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial t} = \frac{\partial}{\partial y}\left(1 + \alpha\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial y} - [\delta(t) + \tau\delta'(t)]u_{ss}(y).$$

4. Now, solve BVP via eigenfunction expansion, $v(y,t) = \sum_n a_n(t)\psi_n(y)$.

- $\approx$ Duhamels principle: time-varying boundary condition "exchanged" for a time-varying source term!

Discussion

- The transformation $u(y,t) = v(y,t) + u_{ss}(y)$
  - alters the solution for $t < 0$, which is supposed to be $u \equiv 0$ (fluid at rest);
  - "condenses" the cumulative effects from $t = 0$ to $t = \infty$ into an initial condition, namely $v(y,0) = -u_{ss}(y)$...violation of causality ("no input before the output").
- The transformation $u(y,t) = v(y,t) + H(t)u_{ss}(y)$
  - respects condition of rest prior to start-up;
  - replaces time-dependent boundary condition with time-dependent source, initial condition is still state of rest ($u \equiv 0$).
- Is there a more general principle at play here?

Summary

- Proposed the correct, causality-satisfying transformation to a homogeneous BVP for start-up problems:

$$\boxed{u \mapsto u + H(t)\, u_{\text{steady-state}}.}$$

- Confirmed with independent Laplace-transform solution and numerical simulations.
- Bold proposal: **revise how we teach this topic?**
- Ref.: I.C. Christov, "On a difficulty in the formulation of initial and boundary conditions for eigenfunction expansion solutions for the start-up of fluid flow," *Mechanics Research Communications* **51** (2013) 86–92; arXiv:1305.5999.

Thank you for your attention!

DMS-1104047

LANL is operated by Los Alamos National Security, L.L.C. for the National Nuclear Security Administration of the U.S. Department of Energy under Contract No. DE-AC52-06NA25396.