Wavelet-Galerkin Methods for Differential Equations

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Matrix Analysis and Wavelets REU

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Overview

- Introduction to the Galerkin Method.
- Brief introduction to Sobolev Spaces and their purpose.
- 2 Dimensional MRAs and Wavelets.
- Improving approximation with scaling functions and wavelets by
  - the Elevation scheme.
  - the Leverage Scheme.
- Wavelet-Galerkin Methods
  - with 1D MRAs and Wavelets.
  - with 2D MRAs and Wavelets.
- Related open problems.
- Concluding remarks.

The Galerkin Method

Consider the following Boundary Value Problem:

$$L[u(x,y)] = 0 \;\; on \;\; D(x,y), \;\; and \;\; S(u) = 0 \;\; on \;\; \partial D \tag{1}$$

and let,

$$u(x,y) \approx u_a(x,y) = u_0(x,y) + \sum_{i=1}^{N} a_i g_i(x,y),$$

then by applying (1),

$$L[\sum_{i=1}^{N} a_i g_i(x,y)] + L[u_0(x,y)] = R(a_1, ..., a_N, x, y)$$

The Galerkin Method (cont'ed)

Minimize $R(a_1, ..., a_N, x, y)$ with respect to $\{g_i\}_{i=1}^N$ as follows,

$$\langle R(a_1, ..., a_N, x, y), g_i(x, y)\rangle_{L^2} = 0, \quad i = 1, 2, ..., N$$

and with some algebra we have,

$$\sum_{j=1}^{N} a_j \langle L[g_j(x, y)], g_i(x, y)\rangle + \langle L[u_a(x, y)], g_i(x, y)\rangle = 0. \qquad (2)$$

The latter can be rewritten in matrix form as follows,

$$\begin{pmatrix} \langle L[g_1], g_1\rangle & \dots & \langle L[g_N], g_1\rangle \\ \vdots & \ddots & \vdots \\ \langle L[g_1], g_N\rangle & \dots & \langle L[g_N], g_N\rangle \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} = - \begin{pmatrix} \langle L[u_a], g_1\rangle \\ \vdots \\ \langle L[u_a], g_N\rangle \end{pmatrix}.$$

Sobolev Spaces

- Definition:
  Let $\Omega \subset \mathbb{R}^k$, then the Sobolev Space of order s on Ω is
  $H^s(\Omega) = \{f(x_1, ...x_N) \in L^2(\Omega) \mid \frac{\partial^k f}{\partial x_j^k} \in L^2(\Omega)\}$
  where $j = 1, ..., N$ and $k = 1, ..., s$.
- Sobolev Norm and Semi-Norm:

$$\|f\|_{s,\Omega}^2 = \sum_{i=0}^{s} \int_\Omega |f^{(i)}|^2 du, \quad |f|_{s,\Omega}^2 = \int_\Omega |f^{(s)}|^2 du$$

- Iterative Inner Product:

$$\langle f, g \rangle_{s,\Omega}^2 = \sum_{i=0}^{s} \int_\Omega \frac{\partial^i f}{\partial x^i} \frac{\partial^i g}{\partial x^i} du$$

Why Sobolev Spaces?

- Guarantee existence of s square-integrable derivatives on the solution interval.
- Allow imposition of boundary conditions on the space. Ex:
  1D Dirichlet BC - $H_0^s(\Omega) = \{f \in H^s(\Omega) \mid f(a) = f(b) = 0\}$
- Functional subspaces of $L^2(\mathbb{R}^k)$, so
  - can be solutions spaces for PDEs.
  - frames for L^2 are frames for H^s.
- In the Sobolev Space of order 1 the semi-norm is a norm.

2D Scaling Functions and Wavelets

- Tensor products of 1D scaling functions and wavelets:
  Scaling Function -

$$\phi_{(n,j),(m,k)} = \phi_{n,j} \otimes \phi_{m,k}$$

Three Wavelets -

$$\psi^1_{(n,j),(m,k)} = \phi_{n,j} \otimes \psi_{m,k},$$

$$\psi^2_{(n,j),(m,k)} = \psi_{n,j} \otimes \phi_{m,k},$$

$$\psi^3_{(n,j),(m,k)} = \psi_{n,j} \otimes \psi_{m,k}$$

- Triorthogonal wavelet system by construction.

2D ... Wavelets (cont'ed)

- Construct a new scaling function and wavelets as,

$$\phi_{n,\vec{k}}(\vec{x}) = \phi(A^{-n}\vec{x} - \vec{k}), \quad \psi_{n,\vec{k}}(\vec{x}) = \phi(A^{-n}\vec{x} - \vec{k})$$

where A is the dilation matrix, typically $const. \cdot I_2$.

- $\{\psi_{n,\vec{k}}\}$ is a basis of $L^2(\mathbb{R}^2)$.
- Orthogonality Condition:

$$For \ \ \psi \in L^2(\mathbb{R}^2), \ \ |det(A)| = 2$$

$$|H(\vec{\omega})|^2 + |H(\vec{\omega} + \pi(\hat{\imath} + \hat{\jmath})|^2 = 1, \ \ H(0) = 1.$$

$H(\vec{\omega}) = \sum_{\vec{k}} h_{\vec{k}} e^{-i\vec{k}\cdot\vec{\omega}}, \ \ h_{\vec{k}}$ are the scaling coefficients.

Elevation Scheme

- Elevation through the 1D functions:
  - Take the elevation of $\phi(x)$ as $\Phi(x) = \int_\alpha^x \phi(t)dt,\ \alpha \in \mathbb{R}$.
  - Construct $\Psi(x)$ in the same way.
  - Using the 1D elevations construct the 2D elevations via a tensor product.
  - Shortcomings: $\Phi(x, y)$ and $\Psi(x, y)$ are only once differentiable in x and y. MRA formulas do not hold between Φ and Ψ.
- Elevation of the 2D functions:
  - For a PDE requiring 2 x and 2 y partial derivatives:

$$\Phi(x, y) = \int_{\beta_1}^{y} \int_{\beta_2}^{y} \int_{\alpha_1}^{x} \int_{\alpha_2}^{x} \phi(u, v) du du dv dv$$

Leverage Scheme

- Construct a new elevation of ϕ, $\Phi(\vec{x}) = \int \phi(\vec{x}) - \phi(\vec{x} - \vec{u})$
- This elevation forms an MRA on $L^2(\mathbb{R}^2)$
- Construct Ψ using the MRA definition.
- For a complete leverage scheme, construct $\tilde{\Phi}$ using the frequency response criterion: $m_0^{\tilde{\Phi}}(\vec{\omega}) = -e^{-i\omega}\overline{m_0^{\Phi}(\vec{\omega} + \pi(\hat{\imath} + \hat{\jmath}))}$.
- Construct $\tilde{\Psi}$ using MRA formulas.
- $\{\Phi, \tilde{\Phi}, \Psi, \tilde{\Psi}\}$ is a biorthogonal system in $L^2(\mathbb{R}^2)$.
- Advantages: MRA formulas can be used to quickly change resolution levels and functional bases. Four mutually orthogonal sets of functional bases to choose from.

Wavelet-Galerkin Method Type 1

- Let $\{\phi_n(x)\}_{n=1}^{\infty}$ be a frame for $L^2((a,b))$, then the elevation $\{\Phi_n\}_{n=1}^{\infty}$ is a frame too.
- Eliminate linearly dependant terms.
- For some $J > 0$, $\{1, \Psi_{j,k} \mid 0 \leq j < J,\ k \in D_j\}$ and $\{1, \Phi_{J,k} \mid 1 - R \leq k \leq 2^J R - 1\}$ are bases of $S_J \subset H^1((0,R))$. $D_j = \{k \in p - R \leq k \leq 2^j R - p\}$.
- Using the tensor product construction, $\Psi_{n,j}^{m,k}$ are bases for $S_{J,K} \subset H^1 \times H^1$.
- Express the solution in terms of the basis, and solve a linear system of equations for the coefficients.

Wavelet-Galerkin Method Type 2

- Take $\{\phi_{n,\vec{k}}(x,y)\}_{n=1}^{\infty}$ to be a frame of $L^2((a,b)\times(c,d))$. Then the elevation or leverage can be constructed and $\{\Phi_{n,\vec{k}}(x,y)\}_{n=1}^{\infty}$ will be a frame as well.
- Eliminate the linearly dependant functions - for some $J > 0$, $\{1, \Psi_{j,(k_1,k_2)} \mid 0 \leq j < J,\ k_1, k_2 \in D_j\}$ and $\{1, \Phi_{J,(k_1,k_2)} \mid 1 - R \leq k_1, k_2 \leq 2^J R - 1\}$ are bases of $S_J \subset H^s((a,b)\times(c,d))$. $D_j = \{k \in p - R \leq k \leq 2^j R - p\}$.
- Depending on the elevation/leverage these functions can be a basis for a solution space for a PDE with up to s derivatives with respect to x or y.
- Express the BVP solution in terms of the basis and solve a system of linear equations to get the coefficients.

Related Open Problems

- Construction of smooth compactly supported orthogonal wavelets in $\mathbb{R}^2$.
- Adapt the elevation/leverage schemes to a non-square boundary.
- Numerical analysis (comparison) of the elevation/leverage schemes in 2 dimensions and classical 2D Galerkin approaches.
- Numerical Quadrature formulas for improved calculation of involved inner products.
- Analysis of the stiffness matrix - reduction of operations, sparsity.