On the Wavelet-Galerkin Solution to the Korteweg-de Vries Equation

Texas A&M University Summer 2004 Matrix Analysis and Wavelets REU

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Overview

- Overview
- The Galerkin method
- Multiresolution analyses and wavelets
- Role of wavelets in the Galerkin method
- Connection coefficients
- The Korteweg-de Vries equation
- The wavelet-Galerking solution to the KdV equation
- Properties of the solution
- Ideas for further research
- References
Consider the following Boundary Value Problem:

\[ \mathcal{L}[u(x, y)] = 0 \text{ on } \Omega, \text{ and } \mathcal{S}[u(x, y)] = 0 \text{ on } \partial \Omega \]

where \( u \in W^{s,2}(\Omega) \subset L^2(\Omega) \); introduce the approximation

\[ u(x, y) \approx u_a(x, y) \equiv u_0(x, y) + \sum_{j=1}^{N} a_j f_j(x, y) , \]

where \( \{f_j(x, y)\}_{j=1}^{N} \) spans an \( N \) dimensional space. We define \( \mathcal{L}[u_a(x, y)] \) as the “residual”,

\[ R(a_1, \ldots, a_N, x, y) \equiv \mathcal{L} \left[ \sum_{j=1}^{N} a_j f_j(x, y) \right] + \mathcal{L} [u_0(x, y)] \]
Minimize $R$ with respect to $\{g_i(x, y)\}_{i=1}^{N}$ as follows,

$$\langle R(a_1, \ldots, a_N, x, y), g_i(x, y) \rangle_{L^2} = 0, \quad \forall i \in \{1, 2, \ldots, N\}$$

and with some algebra we have,

$$\sum_{j=1}^{N} a_j \langle \mathcal{L}[f_j(x, y)], g_i(x, y) \rangle + \langle \mathcal{L}[u_0(x, y)], g_i(x, y) \rangle = 0, \quad \forall i$$

The latter can be rewritten in matrix form as follows,

$$\begin{pmatrix}
\langle \mathcal{L}[f_1], g_1 \rangle & \cdots & \langle \mathcal{L}[f_N], g_1 \rangle \\
\vdots & \ddots & \vdots \\
\langle \mathcal{L}[f_1], g_N \rangle & \cdots & \langle \mathcal{L}[f_N], g_N \rangle
\end{pmatrix}
\begin{pmatrix}
a_1 \\
\vdots \\
a_N
\end{pmatrix}
= -
\begin{pmatrix}
\langle \mathcal{L}[u_a], g_1 \rangle \\
\vdots \\
\langle \mathcal{L}[u_a], g_N \rangle
\end{pmatrix}$$
The set \( \{V_j\}_{j \in \mathbb{Z}} \) is an MRA of \( L^2(\mathbb{R}) \) if

- **Scale Invariance:** \( f(x) \in V_j \iff f(2x) \in V_{j+1} \)
- **Shift Invariance:** \( f(x) \in V_j \iff f(x - k) \in V_j \ \forall k \in \mathbb{Z} \)
- **Completeness:** \( V_j \subset V_{j+1}, \bigcup_{j=-\infty}^{\infty} V_j = L^2(\mathbb{R}), \bigcap_{j=-\infty}^{\infty} V_j = 0 \)
- **Riesz Basis:** \( \exists \phi(x) \in L^2(\mathbb{R}) \) such that, \( \forall j \in \mathbb{Z}, \{\phi_{j,k} \equiv 2^{j/2} \phi(2^j x - k)\}_{k \in \mathbb{Z}} \) is an orthonormal Riesz basis of \( V_j \)

A scaling relation holds: \( \phi_{j,k} = \sum_n c[n] \phi_{j+1,n+2k} \)

We define the orthogonal complement of \( V_j \) in \( V_{j+1} \) as \( W_j \) such that \( W_j \oplus V_j = V_{j+1} \)

The wavelets \( \{\psi_{j,k}\}_{k \in \mathbb{Z}} \) are a Riesz basis of \( W_j \)
MRAs and the Galerkin Method

- The Mallat decomposition algorithm

\[ L^2(\mathbb{R}) = \bigoplus_{j=m}^{+\infty} V_j \bigoplus_{j=-\infty}^{+\infty} W_j \]

- Approximation by projection onto the MRA

\[ PV_j[f] = \sum_{k=-\infty}^{+\infty} a_{j,k} \phi_{j,k} = \sum_{k=-\infty}^{+\infty} a_{j-1,k} \phi_{j-1,k} + \sum_{k=-\infty}^{+\infty} b_{j-1,k} \psi_{j-1,k} \]

- In the Galerkin method we seek the coefficients \( a_{j,k} \) but in general \( a_{j,k} = \langle f, \phi_{j,k} \rangle \), \( b_{j,k} = \langle f, \psi_{j,k} \rangle \)

- Summation is finite for compactly supported wavelets
Connection Coefficients

For an MRA with a compactly supported scaling function $\phi_{j,k}$ it has been shown that one can compute products of the form

$$\Lambda^{d_1, d_2}_l = \int_{-\infty}^{+\infty} \phi^{(d_1)}_{j,0} \phi^{(d_2)}_{j,l} \, dx = \left\langle \phi^{(d_1)}_{j,0}, \phi^{(d_2)}_{j,l} \right\rangle_{L^2}$$

$$\Lambda^{d_1, d_2, d_3}_l = \int_{-\infty}^{+\infty} \phi^{(d_1)}_{j,0} \phi^{(d_2)}_{j,l} \phi^{(d_3)}_{j,m} \, dx = \left\langle \phi^{(d_1)}_{j,0}, \phi^{(d_2)}_{j,l}, \phi^{(d_3)}_{j,m} \right\rangle_{L^2}$$

exactly and a priori.

No need for a quadrature formula

With connection coefficients the wavelet-Galerkin method reduces to solving a system of ODEs
The Korteweg-de Vries Equation

\[ \mathcal{L}[u] = 0 \text{ where } \mathcal{L}[u] \equiv u_t + \alpha uu_x + \beta u_{xxx} \text{ and in a physical context } \alpha = -6, \beta = 1 \]

Why KdV?

An analytic solution exists and is known:
1 soliton solution: \( u(x,t) = 1/\cosh^2(a(x - ct)) \)
2 soliton solution: (more complicated but can be written in closed form)

Other wavelet methods (non-Galerkin) for nonlinear PDEs do not work well for KdV

No nonlinear traveling wave equations solved in the literature (do MRAs offer good solution spaces for the latter?)

One of the “simpler” nonlinear PDEs that model physical phenomena (shallow water waves)
The Wavelet-Galerkin Solution

Assuming a compactly supported scaling function with sufficient smoothness (say Daubechies) at resolution $j = J$ we have

$$\langle \mathcal{L}[u_a], \phi_{J,l} \rangle = \sum_m \dot{a}_{J,m} \delta_l, m + \alpha \sum_m \sum_k a_{J,m} a_{J,k} \Lambda_{m,k,l}^{0,0,1} + \beta \sum_m a_{J,m} \Lambda_{m-l}^{0,3}$$

for the KdV operator.

For all $m, l \leq N$ we have the matrix equation

$$M \dot{A} + (B + C)A = 0$$

Initial condition $A_{t=0} = D$ comes from applying the Galerkin method the initial condition $u(x, 0) = u_i(t)$ but it reduces to $d_l = u_i(l/2^J)$. 
Theorem: Let $\phi_{j,k}$ have compact support, $p$ vanishing moments, and span the approximation space $V_j$, then for any $f \in L^2(\mathbb{R})$ that is uniformly Lipshitz with exponent $\alpha \leq p$, there exists $C > 0$ such that

$$\|f - P_{V_j}[f]\|_{L^2} \leq C2^{-j\alpha}$$

The solution has geometric convergence in the mean with respect to resolution level and smoothness of the solution.

Matrix $C$ is multidiagonal increasing the speed of the numerical algorithms used.

$$C = \text{for } N = 40 \text{ translates}$$
Further Research Ideas

- Develop and adaptive wavelet-Galerkin algorithm (use information at scale \( j \) to improve solution at scale \( j + 1 \)).
- Compare the wavelet-Galerkin method’s accuracy and convergence to classical methods (e.g. polynomials, trigonometric functions, etc).
- Apply the wavelet-Galerkin method to nonlinear PDEs that have highly localized detail in the solution (e.g. Navier-Stokes modeling turbulence).
- Using the connection coefficients study the form of the matrices \( B \) and \( C \) to determine which PDEs are better approximated using the wavelet-Galerkin method.
- Conduct an experimental study of the accuracy of the wavelet-Galerkin method for different compactly supported wavelets.
References


