On the Wavelet-Galerkin Solution to the Korteweg-de Vries Equation

Texas A&M University Summer 2004 Matrix Analysis and Wavelets REU

Ivan Christov

christov@mit.edu

Massachusetts Institute of Technology

Overview

- Overview
- The Galerkin method
- Multiresolution analyses and wavelets
- Role of wavelets in the Galerkin method
- Connection coefficients
- The Korteweg-de Vries equation
- The wavelet-Galerking solution to the KdV equation
- Properties of the solution
- Ideas for further research
- References

The Galerkin Method

Consider the following Boundary Value Problem:

$$\mathcal{L}[u(x,y)] = 0 \text{ on } \Omega, \text{ and } \mathcal{S}[u(x,y)] = 0 \text{ on } \partial\Omega$$

where $u \in \mathbf{W}^{s,2}(\Omega) \subset \mathbf{L}^2(\Omega)$; introduce the approximation

$$u(x,y) \approx u_a(x,y) \equiv u_0(x,y) + \sum_{j=1}^{N} a_j f_j(x,y),$$

where $\{f_j(x,y)\}_{j=1}^{N}$ spans an N dimensional space. We define $\mathcal{L}[u_a(x,y)]$ as the "residual",

$$R(a_1, ..., a_N, x, y) \equiv \mathcal{L}\left[\sum_{j=1}^{N} a_j f_j(x,y)\right] + \mathcal{L}\left[u_0(x,y)\right]$$

The Galerkin Method (II)

Minimize R with respect to $\{g_i(x,y)\}_{i=1}^{N}$ as follows,

$$\langle R(a_1,...,a_N,x,y), g_i(x,y)\rangle_{\mathbf{L}^2} = 0, \quad \forall i \in \{1,2,...,N\}$$

and with some algebra we have,

$$\sum_{j=1}^{N} a_j \langle \mathcal{L}\left[f_j(x,y)\right], g_i(x,y)\rangle + \langle \mathcal{L}\left[u_0(x,y)\right], g_i(x,y)\rangle = 0, \quad \forall i$$

The latter can be rewritten in matrix form as follows,

$$\begin{pmatrix} \langle \mathcal{L}[f_1], g_1\rangle & ... & \langle \mathcal{L}[f_N], g_1\rangle \\ \vdots & \ddots & \vdots \\ \langle \mathcal{L}[f_1], g_N\rangle & ... & \langle \mathcal{L}[f_N], g_N\rangle \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} = - \begin{pmatrix} \langle \mathcal{L}[u_a], g_1\rangle \\ \vdots \\ \langle \mathcal{L}[u_a], g_N\rangle \end{pmatrix}$$

Multiresolution Analyses of $\mathbf{L}^2(\mathbb{R})$

- The set $\{\mathbf{V}_j\}_{j\in\mathbb{Z}}$ is an MRA of $\mathbf{L}^2(\mathbb{R})$ if
  - **Scale Invariance**: $f(x) \in \mathbf{V}_j \iff f(2x) \in \mathbf{V}_{j+1}$
  - **Shift Invariance**: $f(x) \in \mathbf{V}_j \iff f(x-k) \in \mathbf{V}_j \; \forall k \in \mathbb{Z}$
  - **Completeness**: $\mathbf{V}_j \subset \mathbf{V}_{j+1}$, $\overline{\bigcup_{j=-\infty}^{\infty} \mathbf{V}_j} = \mathbf{L}^2(\mathbb{R})$, $\bigcap_{j=-\infty}^{\infty} \mathbf{V}_j = 0$
  - **Riesz Basis**: $\exists\, \phi(x) \in \mathbf{L}^2(\mathbb{R})$ such that, $\forall j \in \mathbb{Z}$, $\{\phi_{j,k} \equiv 2^{j/2}\phi(2^j x - k)\}_{k\in\mathbb{Z}}$ is an orthonormal Riesz basis of $\mathbf{V}_j$
- A scaling relation holds: $\phi_{j,k} = \sum_n c[n]\phi_{j+1,n+2k}$
- We define the orthogonal complement of $\mathbf{V}_j$ in $\mathbf{V}_{j+1}$ as $\mathbf{W}_j$ such that $\mathbf{W}_j \oplus \mathbf{V}_j = \mathbf{V}_{j+1}$
- The wavelets $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ are a Riesz basis of $\mathbf{W}_j$

MRAs and the Galerkin Method

- The Mallat decomposition algorithm

$$\mathbf{L}^2(\mathbb{R}) = \mathbf{V}_m \bigoplus_{j=m}^{+\infty} \mathbf{W}_j = \bigoplus_{j=-\infty}^{+\infty} \mathbf{W}_j$$

- Approximation by projection onto the MRA

$$\mathcal{P}_{\mathbf{V}_j}[f] = \sum_{k=-\infty}^{+\infty} a_{j,k}\phi_{j,k} = \sum_{k=-\infty}^{+\infty} a_{j-1,k}\phi_{j-1,k} + \sum_{k=-\infty}^{+\infty} b_{j-1,k}\psi_{j-1,k}$$

- In the Galerkin method we seek the coefficients $a_{j,k}$ but in general $a_{j,k} = \langle f, \phi_{j,k} \rangle$, $b_{j,k} = \langle f, \psi_{j,k} \rangle$
- Summation is finite for compactly supported wavelets

Connection Coefficients

- For an MRA with a **compactly supported** scaling function $\phi_{j,k}$ it has been shown that one can compute products of the form

$$\Lambda_l^{d_1,d_2} = \int_{-\infty}^{+\infty} \phi_{j,0}^{(d_1)} \phi_{j,l}^{(d_2)} dx = \left\langle \phi_{j,0}^{(d_1)}, \phi_{j,l}^{(d_2)} \right\rangle_{\mathbf{L}^2}$$

$$\Lambda_{l,m}^{d_1,d_2,d_3} = \int_{-\infty}^{+\infty} \phi_{j,0}^{(d_1)} \phi_{j,l}^{(d_2)} \phi_{j,m}^{(d_3)} dx = \left\langle \phi_{j,0}^{(d_1)} \phi_{j,l}^{(d_2)}, \phi_{j,m}^{(d_3)} \right\rangle_{\mathbf{L}^2}$$

  exactly and a priori.
- No need for a quadrature formula
- With connection coefficients the wavelet-Galerkin method reduces to solving a system of ODEs

The Korteweg-de Vries Equation

- $\mathcal{L}[u] = 0$ where $\mathcal{L}[u] \equiv u_t + \alpha u u_x + \beta u_{xxx}$ and in a physical context $\alpha = -6$, $\beta = 1$
- Why KdV?
  - An analytic solution exists and is known:
    - 1 soliton solution: $u(x,t) = 1/\cosh^2(a(x - ct))$
    - 2 soliton solution: (more complicated but can be written in closed form)
  - Other wavelet methods (non-Galerkin) for nonliear PDEs do not work well for KdV
  - No nonlinear traveling wave equations solved in the literature (do MRAs offer good solution spaces for the latter?)
  - One of the “simpler” nonlinear PDEs that model physical phenomena (shallow water waves)

The Wavelet-Galerkin Solution

- Assuming a compactly supported scaling function with sufficient smoothness (say Daubechies) at resolution $j = J$ we have

$$\langle \mathcal{L}[u_a], \phi_{J,l} \rangle = \sum_m \dot{a}_{J,m} \delta_{l,m} + \alpha \sum_m \sum_k a_{J,m} a_{J,k} \Lambda^{0,0,1}_{m,k,l} + \beta \sum_m a_{J,m} \Lambda^{0,3}_{m-l}$$

  for the KdV operator.

- For all $m, l \leq N$ we have the matrix equation

$$\mathbf{M}\dot{\mathbf{A}} + (\mathbf{B} + \mathbf{C})\mathbf{A} = 0$$

- Intial condition $\mathbf{A}_{t=0} = \mathbf{D}$ comes from applying the Galerkin method the initial condition $u(x, 0) = u_i(t)$ but it reduces to $d_l = u_i(l/2^J)$.

The Wavelet-Galerkin Solution (II)

- **Theorem:** Let $\phi_{j,k}$ have compact support, p vanishing moments, and span the approximation space $\mathbf{V}_j$, then for any $f \in \mathbf{L}^2(\mathbb{R})$ that is uniformly Lipshitz with exponent $\alpha \leq p$, $\exists C > 0$ such that

$$\left\| f - \mathcal{P}_{\mathbf{V}_j}[f] \right\|_{\mathbf{L}^2} \leq C 2^{-j\alpha}$$

- The solution has geometric convergence in the mean with respect to resolution level and smoothness of the solution.
- Matrix $\mathbf{C}$ is multidiagonal increasing the speed of the numerical algorithms used.

$\mathbf{C} =$  for $N = 40$ translates

Further Research Ideas

- Develop and adaptive wavelet-Galerkin algorithm (use information at scale j to improve solution at scale $j+1$).
- Compare the wavelet-Galerkin method's accuracy and convergence to classical methods (e.g. polynomials, trigonometric fucntions, etc).
- Apply the wavelet-Galerkin method to nonlinear PDEs that have highly localized detail in the solution (e.g. Navier-Stokes modeling turbulence).
- Using the connection coefficients study the form of the matrices $\mathbf{B}$ and $\mathbf{C}$ to determine which PDEs are better approximated using the wavelet-Galerkin method.
- Conduct an experimental study of the accuracy of the wavelet-Galerkin method for different compactly supported wavelets.

References

- K. Amaratunga, J. R. Williams, “Wavelet-Galerkin Solution of Boundary Value Problems,” *Archives of Computational Methods in Engineering*, **4**, 3, 243-285: 1997.
- A. Boggess, F. Narkovich, *A First Course in Wavelets with Fourrier Analysis*, Prentice Hall: 2001.
- C. Fletcher, *Computational Galerkin Methods*, Springer-Verlag: 1984.
- A. Latto, H. L. Resnikoff, E. Tenenbaum, “The Evaluation of Connection Coefficients of Compactly Supported Wavelets,” *Proceedings of the French-USA Workshop on Wavelets and Turbulence at Princeton Unversity, June 1991*, Springer-Verlag: 1992.
- S. Mallat, *A Wavelet Tour of Signal Processing 2ed*, Academic Press: 1999.