On the Wavelet-Galerkin Solution to the Korteweg-de Vries Equation

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Matrix Analysis and Wavelets REU

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The Galerkin Method

Consider the following Initial-Value Problem:

$$L[u(\vec{x}, t)] = 0 \quad \text{on} \quad \Omega \subset \mathbb{R}^n, \quad \text{and} \quad u(\vec{x}, 0) = u_i(\vec{x}),$$

where $u \in W^{s,2}(\Omega) \subset L^2(\Omega)$; introduce the approximation

$$u(\vec{x}) \approx u_a(\vec{x}) \equiv \sum_{j=1}^{N} a_j(t) f_j(\vec{x}),$$

where $\{f_j(\vec{x})\}_{j=1}^{N}$ spans an $N$ dimensional subspace of $L^2(\Omega)$. We define $L[u_a(\vec{x})]$ as the “residual,”

$$R(a_1, \ldots, a_N, \vec{x}) \equiv L \left[ \sum_{j=1}^{N} a_j(t) f_j(\vec{x}) \right].$$
The Galerkin Method (II)

Then we minimize $R$ with respect to $\{g_i(\vec{x})\}_{i=1}^N \subset L^2(\Omega)$ as follows,

$$\langle R(a_1, \ldots, a_N, \vec{x}), g_i(\vec{x}) \rangle_{L^2} = 0, \quad \forall i \in \{1, 2, \ldots, N\}$$

and with some algebra we have,

$$\sum_{j=1}^{N} \langle L[a_j(t)f_j(\vec{x})], g_i(\vec{x}) \rangle_{L^2} = 0, \quad \forall i.$$ 

Taking $L = L_s + L_t$, i.e. the stationary and time-dependent parts of the operator, we obtain the system of ODE:

$$B L_t[A] + CA = 0,$$

$$A = \{a_i\}, \quad B = \{b_{ij} = \langle f_j, g_i \rangle_{L^2}\}, \quad C = \{c_{ij} = \langle L_s[f_j], g_i \rangle_{L^2}\}.$$
The set \( \{ V_j \}_{j \in \mathbb{Z}} \) is an MRA of \( L^2(\mathbb{R}) \) if

- **Scale Invariance:** \( f(x) \in V_j \iff f(2x) \in V_{j+1} \),

- **Shift Invariance:** \( f(x) \in V_j \iff f(x - k) \in V_j \quad \forall k \in \mathbb{Z} \),

- **Completeness:** \( V_j \subset V_{j+1} \), \( \bigcup_{j=-\infty}^{+\infty} V_j = L^2(\mathbb{R}) \), and \( \bigcap_{j=-\infty}^{+\infty} V_j = 0 \),

- **Riesz Basis:** \( \exists \phi(x) \in L^2(\mathbb{R}) \) such that, \( \forall j \in \mathbb{Z} \),

\( \{ \phi_{j,k} \equiv 2^{j/2} \phi(2^j x - k) \}_{k \in \mathbb{Z}} \) is an orthonormal basis of \( V_j \).

We define the orthogonal complement of \( V_j \) in \( V_{j+1} \) as \( W_j \) such that \( W_j \oplus V_j = V_{j+1} \).

A scaling relation holds: \( \phi_{j,k} = \sum_n c[n] \phi_{j+1,n+2k} \) and \( \psi_{j,k} = \sum_n d[n] \phi_{j+1,n+2k} \) where \( c[n] \) and \( d[n] \) are known.

The wavelet’s translations \( \{ \psi_{j,k} \}_{k \in \mathbb{Z}} \) form a basis of \( W_j \).
The Mallat decomposition algorithm states that

\[ L^2(\mathbb{R}) = \bigoplus_{j=m}^{+\infty} V_j \bigoplus_{j=-\infty}^{j=m} W_j. \]

The Galerkin approximation can be thought of as projection onto a subspace in the MRA:

\[ \mathcal{P}_V[f] = \sum_{k=-\infty}^{+\infty} a_{j,k} \phi_{j,k} = \sum_{k=-\infty}^{+\infty} a_{j-1,k} \phi_{j-1,k} + \sum_{k=-\infty}^{+\infty} b_{j-1,k} \psi_{j-1,k}. \]

In the Galerkin method we seek the coefficients \( a_{j,k} \) but in general we know that \( a_{j,k} = \langle f, \phi_{j,k} \rangle \), \( b_{j,k} = \langle f, \psi_{j,k} \rangle \).

All summation are finite for compactly supported wavelets (no truncation is necessary!).
Connection Coefficients

For an MRA with a compactly supported scaling function $\phi_{j,k}$ it has been shown that one can compute products of the form

$$\Lambda_{d_1,d_2} = \int_{-\infty}^{+\infty} \phi_{j,0}^{(d_1)} \phi_{j,l}^{(d_2)} \, dx = \left\langle \phi_{j,0}^{(d_1)}, \phi_{j,l}^{(d_2)} \right\rangle_{L^2},$$

$$\Lambda_{d_1,d_2,d_3} = \int_{-\infty}^{+\infty} \phi_{j,0}^{(d_1)} \phi_{j,l}^{(d_2)} \phi_{j,m}^{(d_3)} \, dx = \left\langle \phi_{j,0}^{(d_1)}, \phi_{j,l}^{(d_2)}, \phi_{j,m}^{(d_3)} \right\rangle_{L^2}$$

exactly and a priori.

No need for a (approximate) quadrature formula for the integrals.

Connection coefficients corresponding to integrals of products of scaling functions and wavelets can be constructed and found as well.
The Korteweg-de Vries Equation

- **PDE:** \( \mathcal{L}[u(x, t)] \equiv u_t + \alpha uu_x + \beta u_{xxx} = 0; \) in a physical context we take \( \alpha = 6 \) and \( \beta = 1. \)

- **Why KdV?**
  - An analytic solution exists (solitons on an infinite domain):
    - 1-soliton solution: \( u(x, t) = \text{sech}^2(a(x - ct)), a, b \in \mathbb{R}. \)
    - \( M \)-soliton solution: (can be found explicitly).
  - Beylkin’s non-Galerkin wavelet method for nonlinear PDEs leads to dense matrices for wave-like solutions \( \rightarrow \) does not work well for KdV.
  - Test if MRAs offer good solution spaces for nonlinear wave-like solutions.
  - A good candidate of a physically-relevant nonlinear PDE, besides Burger’s Eq., to test the wavelet-Galerkin method on, which also has a wave-like solution (the soliton).
The Wavelet-Galerkin Solution

Assuming a compactly supported scaling function with sufficient smoothness (we take Daubechies’) at a resolution $j = J$ we have

$$\langle R, \phi_{J,l} \rangle = \sum_m \dot{a}_{J,m} \delta_{l,m} + \alpha \sum_m \sum_k a_{J,m} a_{J,k} \Lambda_{m,k,l}^{0,0,1} + \beta \sum_m a_{J,m} \Lambda_{m-l}^{0,3}$$

using the KdV operator.

For all indices $m, l \leq N$ we have the matrix equation

$$M \dot{A} + (B + C) A = 0$$

$$M = \{ \delta_{l,m} \}, \quad B = \left\{ \sum_k a_{J,k} \Lambda_{m,k,l}^{0,0,1} \right\}, \quad C = \left\{ \Lambda_{m-l}^{0,3} \right\}, \quad A = \{ a_i \}$$

The initial condition for the ODEs is $A_i = \{ u_i(l/2^J) \}$
Theorem: Let \( \{ \phi_{j,k} \}_{k \in \mathbb{Z}} \) have compact support, \( p \) vanishing moments, and span the approximation space \( V_j \), then for any \( f \in C^p(\mathbb{R}) \), \( \exists C > 0 \) such that

\[
\| f - \mathcal{P}_{V_j}[f] \|_{L^2} \leq C 2^{jp} \| f^{(p)} \|_{L^2}
\]

Thus, the wavelet-Galerkin solution has geometric convergence in the mean with respect to the resolution level for any sufficiently continuous solution (note: a sech soliton is infinitely differentiable).

Furthermore, in a compactly supported orthonormal wavelet basis, the matrix \( C \) becomes symmetric and multidiagonal, and the matrix \( M \) becomes diagonalized to the identity.
Further Research Ideas

- Develop and adaptive wavelet-Galerkin algorithm (use information at scale $j$ to improve solution at scale $j + 1$).
- Compare the wavelet-Galerkin method’s accuracy and convergence to classical Galerkin methods (e.g. with polynomials or trigonometric bases).
- Apply the wavelet-Galerkin method to nonlinear PDEs that have highly localized detail in the solution (e.g. the Navier-Stokes equations).
- Using the connection coefficients study the form of the matrices $B$ and $C$ to determine which differential operators are best approximated using the wavelet-Galerkin method.
- Conduct an experimental study of the accuracy and convergence of the wavelet-Galerkin method for different compactly supported wavelets.


