On the Wavelet-Galerkin Solution to the Korteweg-de Vries Equation

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Overview

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The Galerkin Method

Consider the following Initial-Value Problem:

$$\mathcal{L}[u(\vec{x},t)] = 0 \ \text{ on } \ \Omega \subset \mathbb{R}^n, \ \text{ and } \ u(\vec{x},0) = u_i(\vec{x}),$$

where $u \in \mathbf{W}^{s,2}(\Omega) \subset \mathbf{L}^2(\Omega)$; introduce the approximation

$$u(\vec{x}) \approx u_a(\vec{x}) \equiv \sum_{j=1}^{N} a_j(t) f_j(\vec{x}),$$

where $\{f_j(\vec{x})\}_{j=1}^N$ spans an N dimensional subspace of $\mathbf{L}^2(\Omega)$. We define $\mathcal{L}[u_a(\vec{x})]$ as the “residual,”

$$R(a_1, ..., a_N, \vec{x}) \equiv \mathcal{L}\left[\sum_{j=1}^{N} a_j(t) f_j(\vec{x})\right].$$

The Galerkin Method (II)

Then we minimize R with respect to $\{g_i(\vec{x})\}_{i=1}^{N} \subset \mathbf{L}^2(\Omega)$ as follows,

$$\langle R(a_1, ..., a_N, \vec{x}), g_i(\vec{x})\rangle_{\mathbf{L}^2} = 0, \quad \forall i \in \{1, 2, ..., N\}$$

and with some algebra we have,

$$\sum_{j=1}^{N} \langle \mathcal{L}\left[a_j(t) f_j(\vec{x})\right], g_i(\vec{x})\rangle_{\mathbf{L}^2} = 0, \quad \forall i.$$

Taking $\mathcal{L} = \mathcal{L}_s + \mathcal{L}_t$, i.e. the stationary and time-dependent parts of the operator, we obtain the system of ODE:

$$\mathbf{B}\mathcal{L}_t[\mathbf{A}] + \mathbf{C}\mathbf{A} = 0,$$

$$\mathbf{A} = \{a_i\}, \quad \mathbf{B} = \{b_{ij} = \langle f_j, g_i\rangle_{\mathbf{L}^2}\}, \quad \mathbf{C} = \{c_{ij} = \langle \mathcal{L}_s[f_j], g_i\rangle_{\mathbf{L}^2}\}.$$

Multiresolution Analyses of $\mathbf{L}^2(\mathbb{R})$

- The set $\{\mathbf{V}_j\}_{j\in\mathbb{Z}}$ is an MRA of $\mathbf{L}^2(\mathbb{R})$ if
  - **Scale Invariance**: $f(x) \in \mathbf{V}_j \iff f(2x) \in \mathbf{V}_{j+1}$,
  - **Shift Invariance**: $f(x) \in \mathbf{V}_j \iff f(x-k) \in \mathbf{V}_j \; \forall k \in \mathbb{Z}$,
  - **Completeness**: $\mathbf{V}_j \subset \mathbf{V}_{j+1}$, $\overline{\bigcup_{j=-\infty}^{+\infty} \mathbf{V}_j} = \mathbf{L}^2(\mathbb{R})$, and $\bigcap_{j=-\infty}^{+\infty} \mathbf{V}_j = 0$,
  - **Riesz Basis**: $\exists\ \phi(x) \in \mathbf{L}^2(\mathbb{R})$ such that, $\forall j \in \mathbb{Z}$, $\{\phi_{j,k} \equiv 2^{j/2}\phi(2^j x - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $\mathbf{V}_j$.
- We define the orthogonal complement of $\mathbf{V}_j$ in $\mathbf{V}_{j+1}$ as $\mathbf{W}_j$ such that $\mathbf{W}_j \oplus \mathbf{V}_j = \mathbf{V}_{j+1}$.
- A scaling relation holds: $\phi_{j,k} = \sum_n c[n]\phi_{j+1,n+2k}$ and $\psi_{j,k} = \sum_n d[n]\phi_{j+1,n+2k}$ where $c[n]$ and $d[n]$ are known.
- The wavelet's translations $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ form a basis of $\mathbf{W}_j$.

MRAs and the Galerkin Method

- The Mallat decomposition algorithm states that

$$\mathbf{L}^2(\mathbb{R}) = \mathbf{V}_m \bigoplus_{j=m}^{+\infty} \mathbf{W}_j = \bigoplus_{j=-\infty}^{+\infty} \mathbf{W}_j.$$

- The Galerkin approximation can be thought of as projection onto a subspace in the MRA:

$$\mathcal{P}_{\mathbf{V}_j}[f] = \sum_{k=-\infty}^{+\infty} a_{j,k}\phi_{j,k} = \sum_{k=-\infty}^{+\infty} a_{j-1,k}\phi_{j-1,k} + \sum_{k=-\infty}^{+\infty} b_{j-1,k}\psi_{j-1,k}.$$

- In the Galerkin method we seek the coefficients $a_{j,k}$ but in general we know that $a_{j,k} = \langle f, \phi_{j,k}\rangle$, $b_{j,k} = \langle f, \psi_{j,k}\rangle$.
- All summation are finite for compactly supported wavelets (no truncation is necessary!).

Connection Coefficients

- For an MRA with a **compactly supported** scaling function $\phi_{j,k}$ it has been shown that one can compute products of the form

$$\Lambda_l^{d_1,d_2} = \int_{-\infty}^{+\infty} \phi_{j,0}^{(d_1)} \phi_{j,l}^{(d_2)} dx = \left\langle \phi_{j,0}^{(d_1)}, \phi_{j,l}^{(d_2)} \right\rangle_{\mathbf{L}^2},$$

$$\Lambda_{l,m}^{d_1,d_2,d_3} = \int_{-\infty}^{+\infty} \phi_{j,0}^{(d_1)} \phi_{j,l}^{(d_2)} \phi_{j,m}^{(d_3)} dx = \left\langle \phi_{j,0}^{(d_1)} \phi_{j,l}^{(d_2)}, \phi_{j,m}^{(d_3)} \right\rangle_{\mathbf{L}^2}$$

  exactly and a priori.
- No need for a (approximate) quadrature formula for the integrals.
- Connection coefficients corresponding to integrals of products of scaling functions and wavelets can be constructed and found as well.

The Korteweg-de Vries Equation

- PDE: $\mathcal{L}[u(x,t)] \equiv u_t + \alpha u u_x + \beta u_{xxx} = 0$; in a physical context we take $\alpha = 6$ and $\beta = 1$.
- Why KdV?
  - An analytic solution exists (solitons on an infinite domain):
    - 1-soliton solution: $u(x,t) = \text{sech}^2(a(x - ct))$, $a, b \in \mathbb{R}$.
    - M-soliton solution: (can be found explicitly).
  - Beylkin's non-Galerkin wavelet method for nonliear PDEs leads to dense matrices for wave-like solutions $\rightarrow$ does not work well for KdV.
  - Test if MRAs offer good solution spaces for nonlinear wave-like solutions.
  - A good candidate of a physically-relevant nonlinear PDE, besides Burger's Eq., to test the wavelet-Galerkin method on, which also has a wave-like solution (the soliton).

The Wavelet-Galerkin Solution

- Assuming a compactly supported scaling function with sufficient smoothness (we take Daubechies') at a resolution $j = J$ we have

$$\langle R, \phi_{J,l} \rangle = \sum_m \dot{a}_{J,m} \delta_{l,m} + \alpha \sum_m \sum_k a_{J,m} a_{J,k} \Lambda^{0,0,1}_{m,k,l} + \beta \sum_m a_{J,m} \Lambda^{0,3}_{m-l}$$

  using the KdV operator.

- For all indices $m, l \leq N$ we have the matrix equation

$$\mathbf{M}\dot{\mathbf{A}} + (\mathbf{B} + \mathbf{C})\mathbf{A} = 0$$

$$\mathbf{M} = \{\delta_{l,m}\}, \quad \mathbf{B} = \left\{ \sum_k a_{J,k} \Lambda^{0,0,1}_{m,k,l} \right\}, \quad \mathbf{C} = \left\{ \Lambda^{0,3}_{m-l} \right\}, \quad \mathbf{A} = \{a_i\}$$

- The initial condition for the ODEs is $\mathbf{A}_i = \{u_i(l/2^J)\}$

The Wavelet-Galerkin Solution (II)

- **Theorem:** Let $\{\phi_{j,k}\}_{k\in\mathbb{Z}}$ have compact support, p vanishing moments, and span the approximation space $\mathbf{V}_j$, then for any $f \in \mathbf{C}^p(\mathbb{R})$, $\exists C > 0$ such that

$$\left\| f - \mathcal{P}_{\mathbf{V}_j}[f] \right\|_{\mathbf{L}^2} \leq C 2^{jp} \| f^{(p)} \|_{\mathbf{L}^2}$$

- Thus, the wavelet-Galerkin solution has geometric convergence in the mean with respect to the resolution level for any sufficiently continuous solution (note: a sech soliton is infinitely differentiable).
- Furthermore, in a compactly supported orthonormal wavelet basis, the matrix $\mathbf{C}$ becomes symmetric **and** multidiagonal, and the matrix $\mathbf{M}$ becomes diagonalized to the identity.

Further Research Ideas

- Develop and adaptive wavelet-Galerkin algorithm (use information at scale j to improve solution at scale $j+1$).
- Compare the wavelet-Galerkin method's accuracy and convergence to classical Galerkin methods (e.g. with polynomials or trigonometric bases).
- Apply the wavelet-Galerkin method to nonlinear PDEs that have highly localized detail in the solution (e.g. the Navier-Stokes equations).
- Using the connection coefficients study the form of the matrices $\mathbf{B}$ and $\mathbf{C}$ to determine which differential operators are best approximated using the wavelet-Galerkin method.
- Conduct an experimental study of the accuracy and convergence of the wavelet-Galerkin method for different compactly supported wavelets.

Principal References

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