A Wavelet Method for Image Anti-Aliasing

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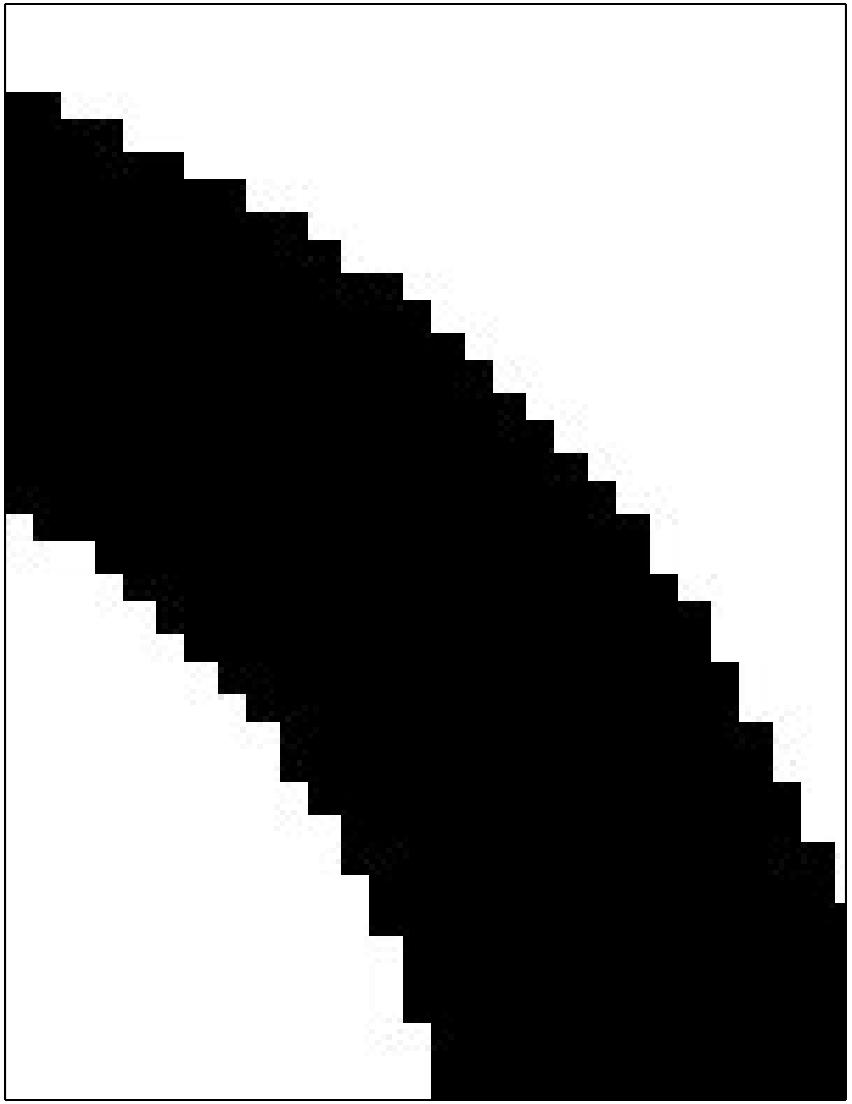
Introduction

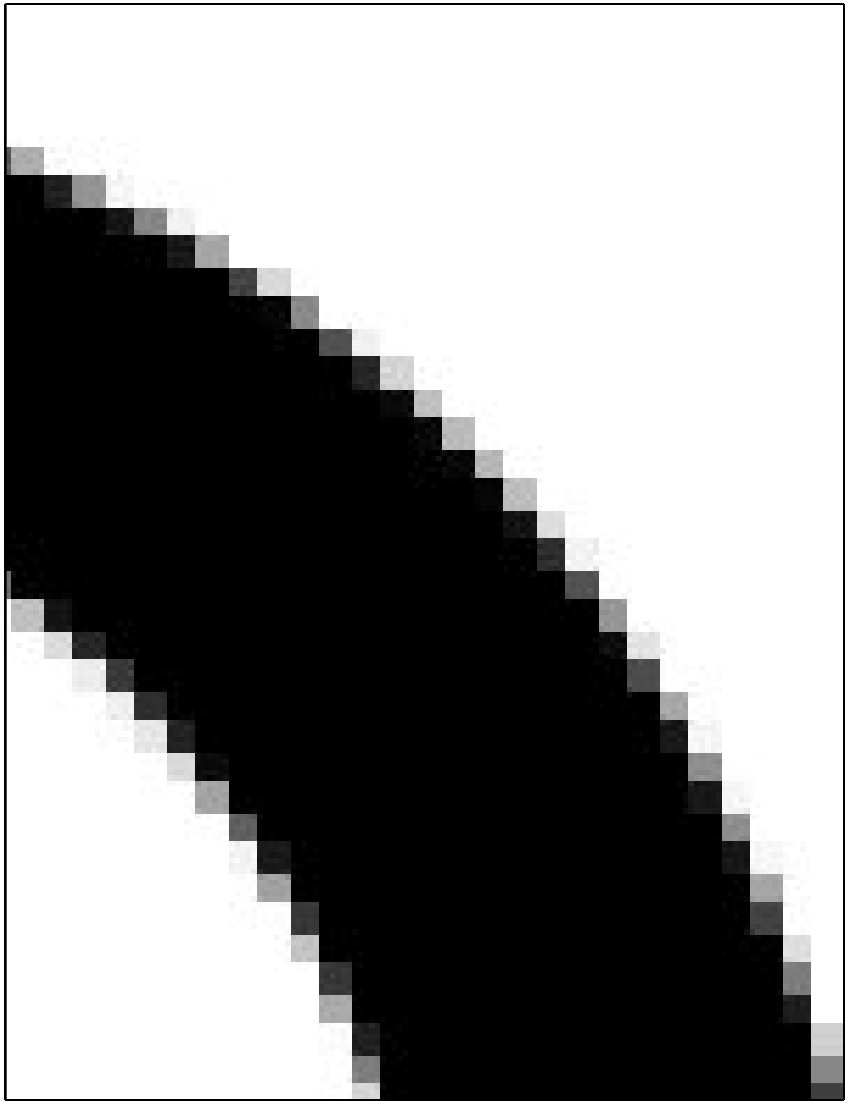
- Aliasing
  - Mathematician's definition: double valued frequency response
  - Engineer's definition: sampling rate below Nyquist
  - Programmer's definition: "jaggies"
- Anti-Aliasing
  - Supersampling/Multisampling
    - nVidia's Intellisample^{TM}– quincunx sampling
    - ATi's SMOOTHVISION^{TM}– jittered sampling
  - Average/filter the discrete Fourier transform (DFT)
  - Supersampling is extravagant, DFT is unforgiving, so why not try a discrete wavelet transform (DWT)?
- Edge Detection

An Example of (Anti-)Aliasing

An Aliased Arc



An Anti–Aliased Arc



Unification of Edges and Aliasing

- The edges of an image $f(x, y)$ are the local maxima of $|\nabla f(x, y)|$
- Mallat proved the edges are also the local maxima of $M_j f(x, y)$ (the modulus of the DWT at resolution j)
- Succinctly stated: Edges $\Leftrightarrow$ large gradient in image intensity $\Leftrightarrow$ large gradient in the frequency domain $\Leftrightarrow$ high frequencies are present $\Leftrightarrow$ aliasing if the resolution is too low
- The DWT's space-frequency localization allows
  - localization of edges
  - localization of aliasing
  - filtering only along edges to remove aliasing

The Mallat-Zhong DWT

- Quadratic spline scaling function and wavelet generated with

$$h[n] = \{\ldots, 0, 1, [3], 3, 1, 0, \ldots\} \cdot \sqrt{2}/4$$

  Short filter $\longrightarrow$ faster computation of convolutions

- Decomposition filters:

$$\bar{h}_j[n] = (\uparrow j)h[-n]$$

$$\bar{g}_j\left[-2^{j-1}\right] = \sqrt{2}/2,\ \bar{g}_j\left[2^{j-1}\right] = -\sqrt{2}/2$$

$$\bar{g}_0[0] = \bar{g}_0[-1] = -\sqrt{2}/2$$

- Reconstruction filters:

$$h_j[n] = \bar{h}_j[-n],\ g_j[n] = \bar{g}_j[-n]$$

The Mallat-Zhong DWT (II)

- The DWT is performed via cascade of separable convolutions (aka filter bank). At each resolution level j

$$\text{Image Coefficients} \quad a_{j+1}[n] = a_j \star \bar{h}_j \bar{h}_j[n]$$

$$\text{Detail Coefficients} \begin{cases} d^1_{j+1}[n] = a_j \star \bar{g}_j \delta[n] \\ d^2_{j+1}[n] = a_j \star \delta \bar{g}_j[n] \end{cases}$$

- And the inverse DWT filter bank is implemented as

$$\tilde{a}_j[n] = \tilde{a}_j \star h_j h_j[n] + d^1_{j+1} \star g_j \delta[n] + d^2_{j+1} \star \delta g_j[n]$$

- M-Z DWT does not have perfect reconstruction (PR) – diagonal detail coefficients are not calculated since they are not necessary for detecting edges
  - Fast Multiscale Edge Detector

Edges and Tresholds

- The modulus of the M-Z DWT at each resolution j is $M_j f[n] = \left|d_j^1[n]\right|^2 + \left|d_j^2[n]\right|^2$, $n = (x, y)$
- The modulus maxima points $\{u_{j,p}\}_p$ are such that $M_j f[u_{j,p}]$ is larger than its two neighbors whose DWT angle (the angle of the vector $(d_j^1[u_{j,p}], d_j^2[u_{j,p}])$) is closest to the angle at $u_{j,p}$.
- Treshold: keep only $u_{j,p}$ such that $M_j f[u_{j,p}] \geq T$, $T \in \mathbb{R}$
  - T usually determined empirically
  - For a noisy signal a better choice is the "SURE" [a] treshold $T = \sigma\sqrt{2 \ln N}$ for an $N \times N$ image, σ is a "robust estimator"

[a] Stein Unbiased Risk Estimator

An Example

Original Image

Modulus Maxima at Resolution 2^1

Modulus Maxima at Resolution 2^1 Tresholded at 0.0084

Modulus Maxima at Resolution of 2^2 Tresholded at 0.0095

Modulus Maxima at Resolution of 2^3 Tresholded at 0.0055

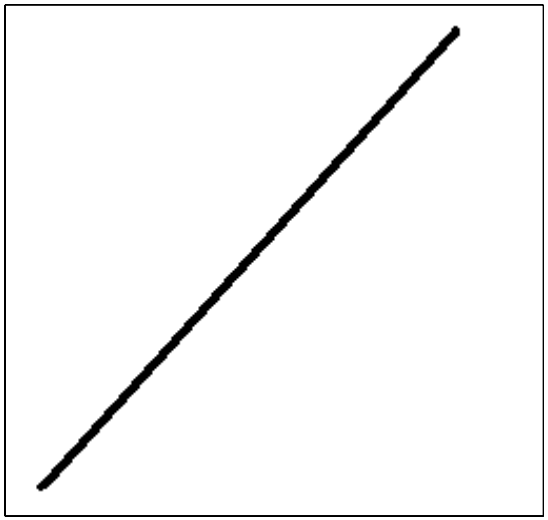
Modulus Maxima at Resolution 2^4 Tresholded at 0.0084

Anti-Aliasing from Edges

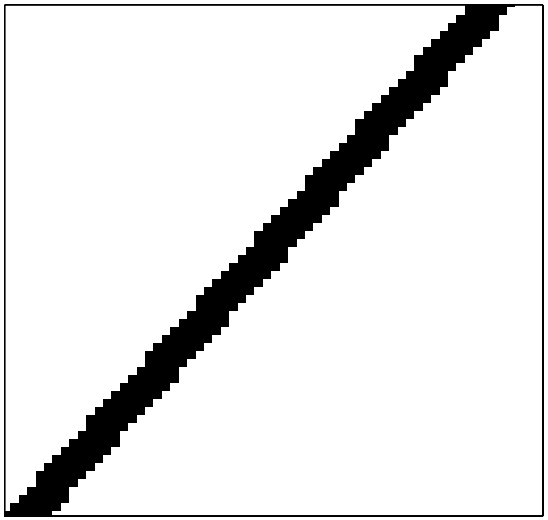
- M-Z DWT inherently provides a global quadratic average of the image (vs. taking a DFT and then averaging)
- At each resolution level $j \leq J$ rescale every $d_j^k[u_{j,p}]$ such that:
  - $\tilde{d}_j^k[u_{j,p}] = R \cdot d_j^k[u_{j,p}]$, $k = \{1, 2\}$
  - $\left|\tilde{d}_j^1[n]\right|^2 + \left|\tilde{d}_j^2[n]\right|^2 \leq T$
- R determined empirically – implementing an adaptive scheme such that $R = R(u_{j,p}, T)$ could improve the results greatly
- Iterate the M-Z DWT with $T = 0$ and $R = 1$ for improved results

Some Pictures

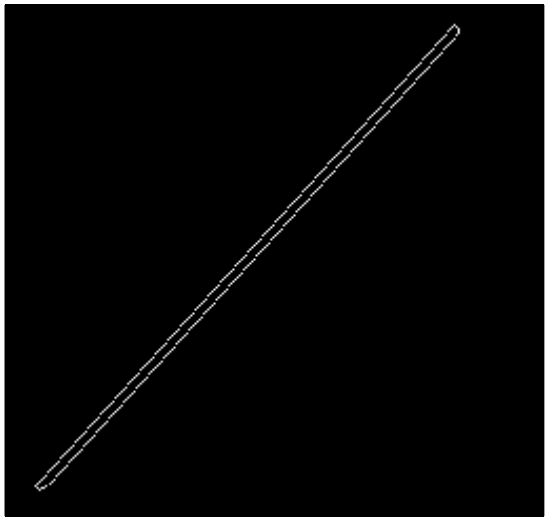
An Aliased Line



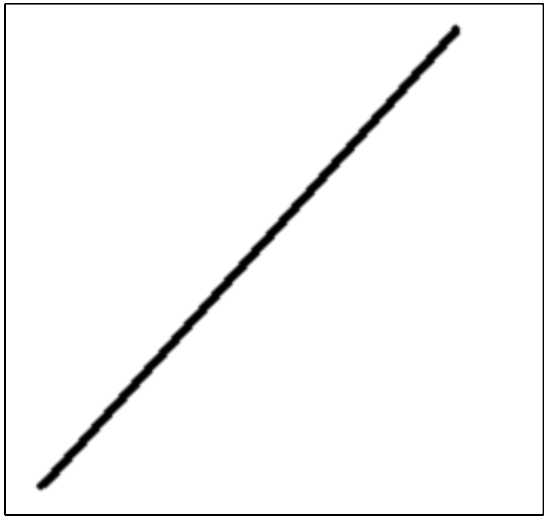
An Aliased Line



Edges of an Aliased Line



An Anti−Aliased with T = 0.5 and R = 0.01



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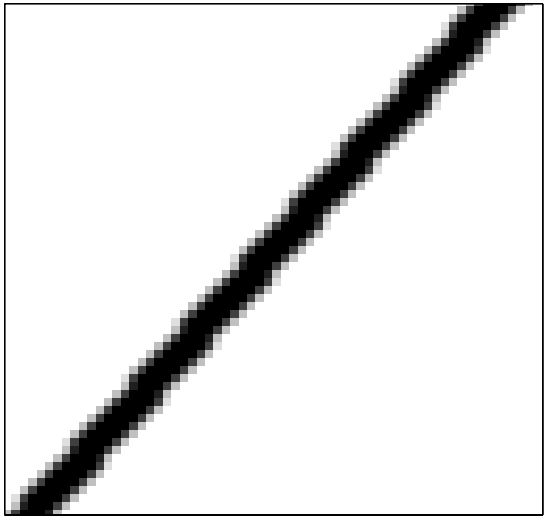
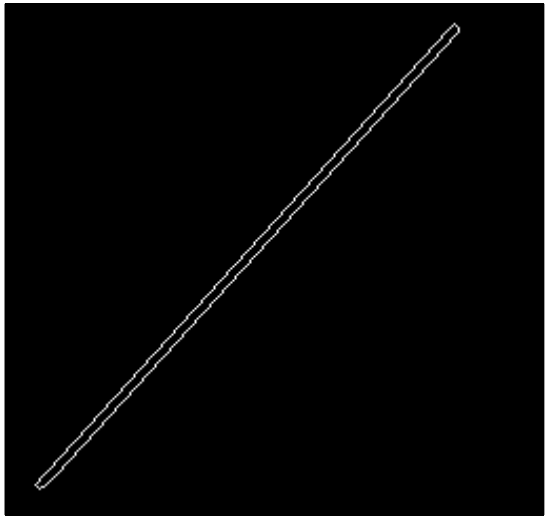
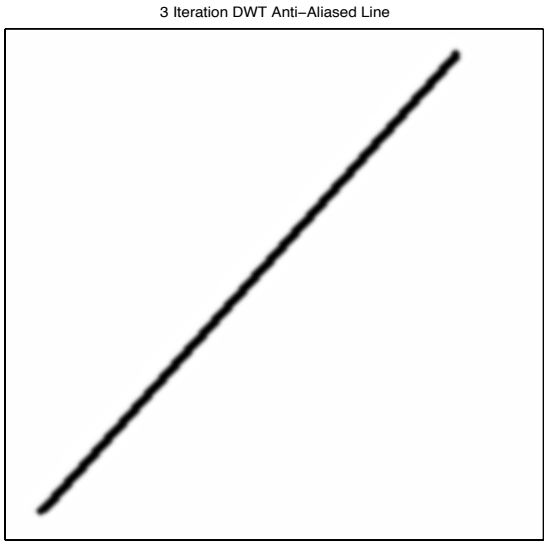
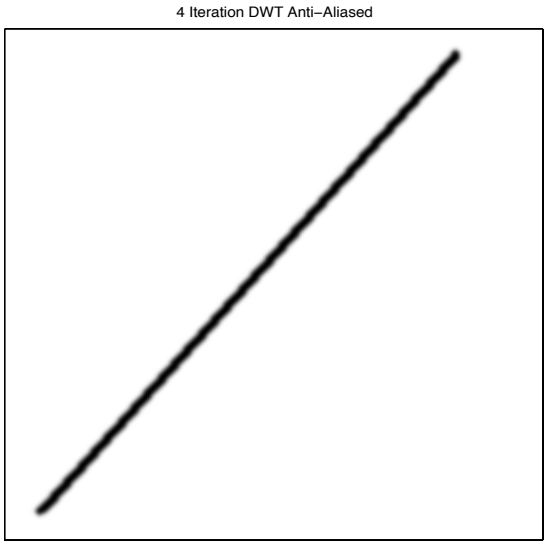


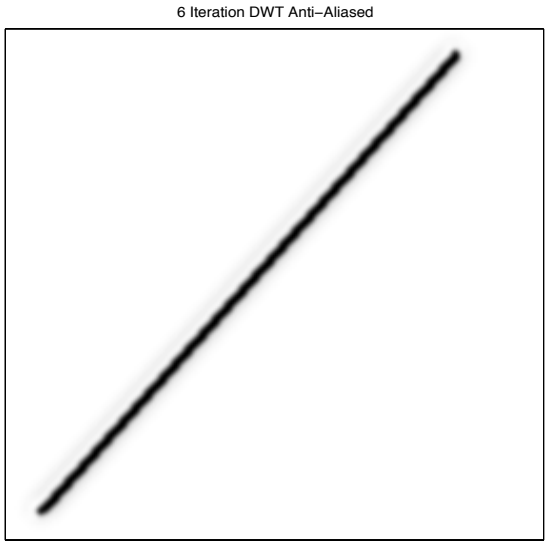
Edges of an Anti−Aliased Line

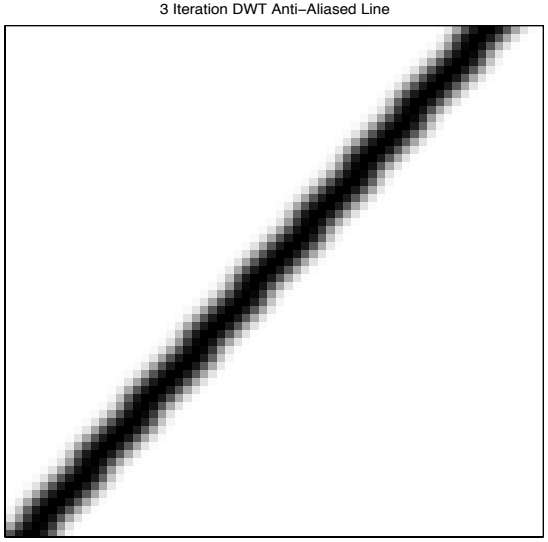


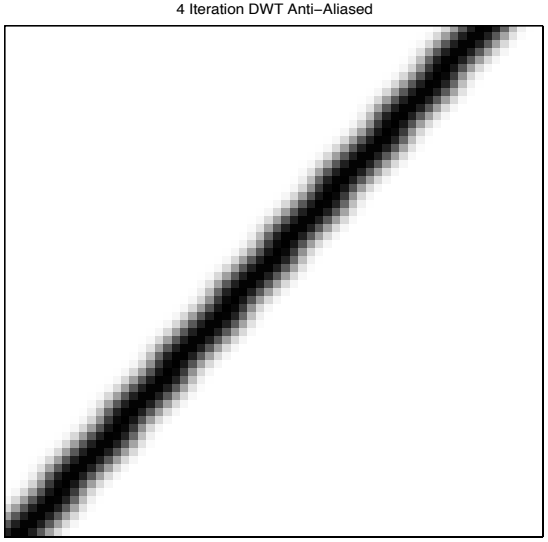
Iterating the DWT

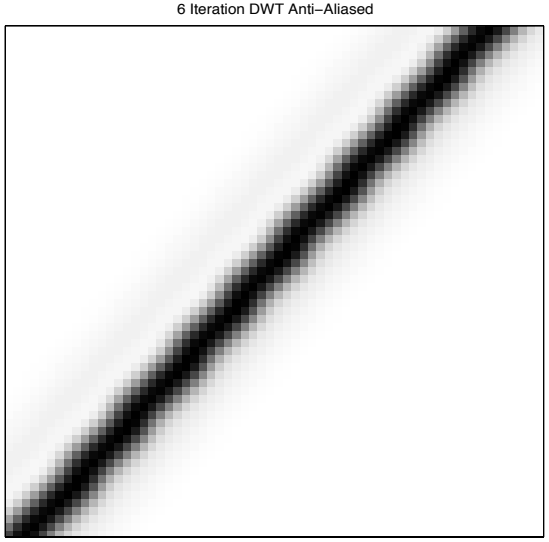












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The End

References

[1] F. Crowe, “The Aliasing Problem in Computer-Generated Shaded Images," *Graphics and Image Processing,* Communications of the AMC: 1977.

[2] S. Mallat, *A Wavelet Tour of Signal Processing 2ed*, Academic Press: 1999.

[3] G. Strang, T. Nguyen, *Wavelets and Filter Banks*, Wellesley-Cambridge Press: 1996.

[4] ATi Technologies, *SMOOTHVISION^{TM} White Paper*, `http://www.ati.com/products/pdf/smoothvision.pdf` : April 16, 2004.

[5] NVIDIA Corporation, *Intellisample Technology Technical Brief*, `http://www.nvidia.com/object/techbrief_hraa.html`, *HRAA: High Resolution Antialiasing through Multisampling Technical Brief*, `http://www.nvidia.com/object/intellisample_tb.html` : April 16, 2004.