Waves in a Shock Tube

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Abstract. This paper discusses linear-wave solutions and simple-wave solutions to the Navier–Stokes equations for an inviscid and compressible fluid in one spatial dimension and one time dimension. We derive wave solutions and study their validity for the initial-boundary-value problem of a shock tube. We solve the initial-boundary-value problem in three cases: a receding piston, an advancing piston, and an oscillating piston.

1. Introduction. The shock-tube problem is often encountered in engineering applications involving pistons. A *shock tube* is a pipe with a moving boundary, such as a piston, and fluid on one side of the boundary. In this paper, we describe the one-dimensional wave motion of the fluid in the tube as the piston compresses or rarefies the fluid.

In particular, we discuss the creation and propagation of shock waves. *Shock waves* are strong discontinuities in the density profile and the velocity profile of the fluid. In order to study the shock waves, we assume the fluid is inviscid, because viscosity dissipates shocks. Furthermore, we assume the fluid is compressible, because compressibility is required for waves to propagate.

Specifically, this paper considers the problem illustrated schematically in Figure 1-1. In this illustration, $u(x,t)$ is the velocity profile, $\rho(x,t)$ is the density profile, and U_p is the piston speed.

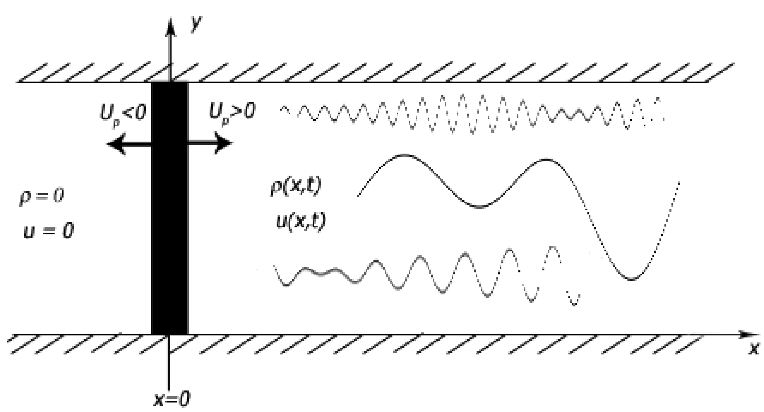

Figure 1-1. Schematic of the problem.

We make the following assumptions:

1. The tube is infinite in the x-direction; so we can neglect reflections of waves at the ends.

2. The flow is independent of the y-position in the tube.
3. The piston is originally at $x = 0$, and is infinitely thin, contrary to what Figure 1-1 shows.
4. The fluid is present only on the right side of the piston, that is, for $x > 0$.
5. The fluid is compressible, inviscid, and isentropic, in other words, an ideal gas.
6. No external forces act on the fluid.

In Section 2, we derive the governing equations from the Navier–Stokes equations of fluid motion. In Section 3, we propose a linear-wave solution and a simple-wave solution to the governing equations of Section 2. In Sections 4 and 5, we find the solution for a piston that only recedes or only advances, respectively. Furthermore, in Section 4, we derive a condition on the velocity of the piston that guarantees the validity of the simple-wave solution. Finally, in Section 6, we derive the complete solution for the specific problem of a piston that advances into the fluid then recedes from it. Sections 3–5 are based on results from Whitham's book [**4**, pp. 161–181], Coulson and Jeffrey's book [**2**, pp. 202–207], and from Landau and Lifshitz's book [**3**, pp. 310–398].

2. The Equations of Gas Dynamics. In this section, we derive the governing equations for the problem, often called the *equations of gas dynamics*. We proceed by simplifying the Navier–Stokes equations using the assumptions made in the introduction.

Consider a compressible fluid with velocity field $\mathbf{u} = (u_x, u_y, u_z)$, density $\rho(x, y, z, t)$, pressure $P(x, y, z, t)$, external force field $\mathbf{F} = (F_x, F_y, F_z)$ and viscosity μ, given in a consistent set of units. For this fluid, the dimensional Navier–Stokes equations are

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla P + \mu\nabla^2\mathbf{u} + \mathbf{F},$$
$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\mathbf{u}).$$

Due to Assumption 2 of Section 1, the velocity field is of the form $\mathbf{u} = (u(x,t), 0, 0)$, and the density is of the form $\rho = \rho(x,t)$. Due to Assumption 5, the viscosity vanishes: $\mu = 0$. Due to Assumption 7, the external force field vanishes: $\mathbf{F} = 0$. Hence, the Navier–Stokes equations reduce to

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} &= 0, \\ \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} &= 0. \end{aligned} \tag{2-1}$$

We have derived two governing equations in three unknowns. To obtain two equations in two unknowns, we must have a constituent relation between two of the three variables. By Assumption 5 of Section 1, the fluid is ideal. It is known that the entropy of an ideal gas is constant, hence a relation exists between the pressure and the density. In the literature, see [**4**, p. 168] for example, the relation is known as the *isentropic gas relation*, and takes the form

$$P = P_0\left(\frac{\rho}{\rho_0}\right)^\gamma$$

for some real number $\gamma > 0$. When the gas is not ideal, we need a third equation, stating the conservation of energy. However, we do not deal with this case.

The isentropic relation is a power law; so the pressure and density must be nondimensionalized in order for the relation to be dimensionally correct. Hence we introduce P_0 and ρ_0, which are the initial pressure and initial density respectively, in order to nondimensionalize the variables.

For simplicity, we introduce the notation $K = \sqrt{\gamma P_0/\rho_0^\gamma}$. Then the isentropic relation implies

$$\frac{\partial P}{\partial x} = K^2 \rho^{\gamma-1} \frac{\partial \rho}{\partial x}.$$

Substituting the preceeding relation into Equations (2-1), the equations of gas dynamics for an isentropic gas become

$$\begin{aligned} u_t + u u_x + \frac{a^2(\rho)}{\rho} \rho_x &= 0, \\ \rho_t + u \rho_x + \rho u_x &= 0 \end{aligned} \tag{2-2}$$

where $a(\rho) = K\rho^{(\gamma-1)/2}$ and the subscripts denote partial derivatives.

In general, the system of PDEs (2-2) cannot be solved analytically; so simplifying assumptions need to be made. In the following section, we first consider the case of linear waves as solutions and then the case of simple waves.

3. Linear-Wave and Simple-Wave Solutions. We begin this section by deriving the governing Equation (3-5) for linear waves from Equations (2-2). Our approach loosely follows the derivation given in Yuan's book [**5**, pp. 411–415].

To derive the linear-wave solution, we assume the density is perturbed slightly from its initial value of ρ_0 by a disturbance $\sigma(x,t)$; that is,

$$\rho(x,t) = \rho_0 \cdot (1 + \sigma(x,t)). \tag{3-1}$$

Physically, this situation corresponds to the piston's moving into the fluid very slowly and over a very short distance, then stopping. For a small perturbation $\sigma(x,t)$, we make the approximation

$$\frac{a^2(\rho)}{\rho} \approx \frac{a^2(\rho_0)}{\rho_0}. \tag{3-2}$$

Differentiating (3-1), then substituting the result and (3-2) into (2-2), we get

$$\begin{aligned} u_t + u u_x + \frac{a^2(\rho_0)}{\rho_0} \rho_0 \sigma_x &= 0, \\ \rho_0 \sigma_t + u \rho_0 \sigma_x + \rho_0 (1+\sigma) u_x &= 0. \end{aligned}$$

Since we are seeking a linear solution, we now linearize these equations by dropping all nonlinear terms and rearranging. We get

$$u_t = -a^2(\rho_0)\sigma_x, \tag{3-3}$$

$$u_x = -\sigma_t. \tag{3-4}$$

By applying $\frac{\partial}{\partial x}$ to Equation (3-3) and $\frac{\partial}{\partial t}$ to Equation (3-4), we obtain two expressions for the mixed second partial derivatives of u. Assuming these derivatives exist and are continuous, which must be true for a small perturbation, the two expressions must be equal. Hence the propagation of a small perturbation $\sigma(x,t)$ in the gas is described by the governing equation

$$\sigma_{tt} = a_0^2 \sigma_{xx} \tag{3-5}$$

where $a_0 = a(\rho_0)$.

The general solution to Equation (3-5) is of the form

$$\sigma(x,t) = F(x - a_0 t) + G(x + a_0 t),$$

namely, two wave forms traveling in opposite directions with velocity $a_0 = K\rho_0^{(\gamma-1)/2}$; see [**1**, p. 142] for a derivation. The functions F and G are determined by the initial conditions and the boundary conditions. Thus we have obtained the complete solution of the problem in terms of linear waves. The existence of the linear-wave solution implies that the shock-tube problem is well defined.

Let us turn to several important concepts exemplified in the linear-wave solution. The collection the lines $\xi = x \pm a_0 t$ for all $\xi \in \mathbb{R}$ are called the *characteristic curves*, or *characteristics*, of the equation. Clearly, the initial wave forms F and G propagate along the characteristics. Moreover, by writing the equations of gas dynamics in the form of Equation (2-2), we anticipated the importance of the function $a(\rho)$. It gives the *speed of sound* in the fluid as a function of the density.

Consider a perturbation $\sigma(x,t)$ traveling with speed a_0. The perturbation can reach only points within its area of influence: $|x| < a_0 t$. Hence, at the interface $|x| = a_0 t$, the density changes from $\rho \cdot (1 + \sigma(x,t))$ to 0. In general, $\sigma(x,t)$ is not 0 at the interface, and therefore a shock wave, or discontinuity in ρ and u, forms for any initial conditions on $\sigma(x,t)$. However, as we show in Section 5, a shock does not always form; thus the linear-wave solution is a poor approximation. So let us consider the simple-wave solutions to Equations (2-2).

Using the simple-wave formulation from Coulson and Jeffrey's book [**2**, p. 200], we take into account the nonlinear terms in the governing equations. To find the simple-wave solutions, we must find the *Riemann invariants* of the system. Whitham, in his book [**4**, pp. 167–170], derives the invariants using the extra assumption $\rho = \rho(u)$; however, we do not make this assumption. Following [**2**, pp. 195–200], we derive the invariant by decoupling the equations of gas dynamics. The decoupling approach, achieved through diagonalization, is both more general and more elegant.

Let us express Equations (2-2) as a matrix equation $\mathbf{Q}_t + \mathbf{B}\mathbf{Q}_x = 0$ where the subscripts denote partial derivatives. Then the matrix equation is as follows:

$$\begin{pmatrix} u \\ \rho \end{pmatrix}_t + \begin{pmatrix} u & \frac{a^2(\rho)}{\rho} \\ \rho & u \end{pmatrix} \begin{pmatrix} u \\ \rho \end{pmatrix}_x = 0. \tag{3-6}$$

To decouple the system, we must diagonalize $\mathbf{B}$ as $\mathbf{B} = \mathbf{P}\Lambda\mathbf{P}^{-1}$ where Λ is a diagonal matrix of the eigenvalues of $\mathbf{B}$ and where $\mathbf{P}$ is the matrix whose columns are the corresponding eigenvectors.

The eigenvalues λ_1 and λ_2 of $\mathbf{B}$ are the roots of the characteristic polynomial:

$$\lambda_1,\ \lambda_2 = u \pm a(\rho).$$

The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} a(\rho) \\ \rho \end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix} -a(\rho) \\ \rho \end{pmatrix},$$

and we can construct $\mathbf{P}$ from them.

Rewriting Equation (3-6) with the diagonalized matrix, we obtain

$$\begin{pmatrix} u \\ \rho \end{pmatrix}_t + \begin{pmatrix} a(\rho) & -a(\rho) \\ \rho & \rho \end{pmatrix} \begin{pmatrix} u + a(\rho) & 0 \\ 0 & u - a(\rho) \end{pmatrix} \begin{pmatrix} a(\rho) & -a(\rho) \\ \rho & \rho \end{pmatrix}^{-1} \begin{pmatrix} u \\ \rho \end{pmatrix}_x = 0.$$

Let us multiply both sides of the preceding equation by $(1/\rho) \cdot \mathbf{P}^{-1}$; we obtain

$$\begin{pmatrix} 1 & \frac{a(\rho)}{\rho} \\ -1 & \frac{a(\rho)}{\rho} \end{pmatrix} \begin{pmatrix} u \\ \rho \end{pmatrix}_t + \begin{pmatrix} u + a(\rho) & 0 \\ 0 & u - a(\rho) \end{pmatrix} \begin{pmatrix} 1 & \frac{a(\rho)}{\rho} \\ -1 & \frac{a(\rho)}{\rho} \end{pmatrix} \begin{pmatrix} u \\ \rho \end{pmatrix}_x = 0.$$

Recalling that $a(\rho) = K\rho^{(\gamma-1)/2}$, we can rewrite the last equation as follows by carrying out the matrix multiplications:

$$\begin{pmatrix} u + \frac{2K}{\gamma-1}\rho^{(\gamma-1)/2} \\ -u + \frac{2K}{\gamma-1}\rho^{(\gamma-1)/2} \end{pmatrix}_t + \begin{pmatrix} u + a(\rho) & 0 \\ 0 & u - a(\rho) \end{pmatrix} \begin{pmatrix} u + \frac{2K}{\gamma-1}\rho^{(\gamma-1)/2} \\ -u + \frac{2K}{\gamma-1}\rho^{(\gamma-1)/2} \end{pmatrix}_x = 0.$$

In turn, we can rewrite the last equation as

$$\begin{pmatrix} -u - \frac{2}{\gamma-1}a(\rho) \\ -u + \frac{2}{\gamma-1}a(\rho) \end{pmatrix}_t + \begin{pmatrix} u + a(\rho) & 0 \\ 0 & u - a(\rho) \end{pmatrix} \begin{pmatrix} -u - \frac{2}{\gamma-1}a(\rho) \\ -u + \frac{2}{\gamma-1}a(\rho) \end{pmatrix}_x = 0. \qquad \text{(3-7)}$$

Thus we arrive at a modified matrix equation $\widetilde{\mathbf{Q}}_t + f(\widetilde{\mathbf{Q}})_x = 0$. Equation (3-7) is known in the literature as the *conservation form* of Equations (2-2).

Let us introduce the following change of variables:

$$v = -u - \frac{2}{\gamma - 1}a(\rho) \text{ and } w = -u + \frac{2}{\gamma - 1}a(\rho). \qquad \text{(3-8)}$$

Then Equation (3-7) becomes

$$\begin{pmatrix} v \\ w \end{pmatrix}_t + \begin{pmatrix} u + a(\rho) & 0 \\ 0 & u - a(\rho) \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_x = 0. \qquad \text{(3-9)}$$

The transformed variables v and w in Equation (3-9) are called the *Riemann invariants* of the system. We can now apply the method of characteristics, described in detail by [**4**, pp. 113–142] and [**2**, pp. 184–191], to each PDE in Equation (3-9).

As in the linear-wave case, we know that the Riemann invariants travel along the characteristics with wave velocity given by the coefficient of the space derivative term in Equation (3-9). However, there are nonlinear contributions now, and the density waves and velocity waves cannot be directly determined by the initial conditions and the boundary conditions. In other words, by the method of characteristics, we know that the quantity v is invariant along characteristics with slope $u + a(\rho)$, and the quantity w is invariant along characteristic with slope $u - a(\rho)$; then, from the values of the invariants on these curves, we can find the density waves and the velocity waves.

We can now derive the solutions of Equations (2-2) along the characteristic curves by making the simple-wave assumption that the Riemann invariants are *constant* for all space and time. Let us label Γ_+ the set of characteristics on which v propagates, and label Γ_- the set on which w propagates. Then v and w are constant on Γ_+ and Γ_- respectively. Hence, we must solve for the set $\Gamma_\pm$. From Equations (3-9), the characteristics $\Gamma_\pm$ are the solutions to

$$\frac{dx}{dt} = u \pm a(\rho) \tag{3-10}$$

with initial condition $x(0) = \xi$ where ξ is the index for the characteristics. Hence, by computing v and w on

$$\Gamma_\pm := \big\{ x(t) = \big(u \pm a(\rho)\big)t + \xi \;\big|\; \xi \in \mathbb{R} \big\},$$

we can obtain the complete solution to Equations (2-2).

4. A Receding Piston. In this section, we consider the waves generated by the motion of a receding piston. Let us consider the case in which the piston in Figure 1-1 moves to the left with velocity $U_p(t) = X_p'(t)$. Furthermore, let us assume that $X_p(t)$ is a "well-behaved" function; in other words, $X_p(t)$ is monotonically decreasing and negative for all times $t \geq 0$. Let us also assume $X_p(0) = 0$; in other words, the piston starts at the origin. Furthermore, we need only consider the right-moving characteristics Γ_+ since no information propagates from the fluid at rest towards the piston.

The initial-boundary-value problem is Equation (2-2) with the following two initial conditions and one boundary condition:

$$\begin{aligned} \rho(x,0) &= \rho_0, \\ u(x,0) &= 0, \\ u(X_p(t),t) &= X_p'(t). \end{aligned} \tag{4-1}$$

Equation (3-9) implies that v is constant on Γ_+. Hence, using the initial conditions, we obtain from Equation (3-8) that

$$v(x,0) = -u(x,0) - \frac{2}{\gamma-1}a(\rho(x,0)) = 0 - \frac{2}{\gamma-1}a_0.$$

Because v is constant, we know that $v(x,t) - v(x,0) = 0$, and hence

$$u - \frac{2}{\gamma-1}\big(a(\rho) - a_0\big) = 0 \tag{4-2}$$

on the Γ_+ characteristics.

To find the density and velocity from the Riemann invariant, we must find the Γ_+ characteristic curves. Let us denote the unknown density near the piston by ρ_{piston}. Then, by integrating Equation (3-10), we find that

$$x(\tau) = \big(X_p(t) + a(\rho_{\text{piston}})\big)(t-\tau) + X_p(\tau)$$

are the characteristics near the piston. Notice that these characteristics are indexed by a time variable τ rather than a spatial variable ξ. The time index τ arises from the boundary condition; that is, we solve Equation (3-10) subject to $x(\tau) = X_p(\tau)$, not $x(0) = \xi$.

Using Equation (4-2) to solve for $a(\rho_{\text{piston}})$, we obtain

$$x(\tau) = \Big(X_p'(\tau) + \frac{\gamma-1}{2}X_p'(\tau) + a_0\Big)(t-\tau) + X_p(\tau).$$

Thus $u(x,t) = X_p'(t)$ and $\rho(x,t) = \rho_{\text{piston}}(x,t)$ along the characteristics:

$$x(\tau) = \Big(\frac{\gamma+1}{2}X_p'(\tau) + a_0\Big)(t-\tau) + X_p(\tau). \tag{4-3}$$

In addition, let us solve for the first characteristic that leaves the piston, since we need this result later. The first characteristic leaves at $\tau = 0$. By assumption, $X_p(0) = 0$; hence,

$$x_{\text{first}}(t) = \Big(a_0 + \frac{\gamma+1}{2}X_p'(0)\Big)t. \tag{4-4}$$

Let us consider the boundary condition at the piston. Then, Equations (4-1) imply that the invariant in Equation (4-2) is

$$X_p'(\tau) - \frac{2}{\gamma-1}\big(a(\rho) - a_0\big) = 0, \quad \text{or} \quad a(\rho) = a_0 + \frac{\gamma-1}{2}X_p'(\tau).$$

We know that $a(\rho) = K\rho^{(\gamma-1)/2}$. Therefore, we have an implicit equation for the density $\rho(x,t)$ near the piston. We can solve this implicit equation, and get

$$\rho_{\text{piston}}(x,t) = \Big(\frac{a_0}{K} + \frac{\gamma-1}{2K}X_p'(\tau)\Big)^{2/(\gamma-1)} \tag{4-5}$$

where $\tau(x,t)$ is implicitly determined by Equation (4-3).

We proceed to the region of space-time not yet affected by the piston. There the density $\rho(x,t)$ is ρ_0 and the velocity $u(x,t)$ is 0, as given by the initial condition. Thus the characteristic curves are

$$x = a_0 t + x_0 \tag{4-6}$$

for any $x_0 \geq 0$.

We do not yet have the complete space-time distribution of velocity and density, because there are characteristics not yet considered. Note that $dx_{\text{first}}/dt > a_0 t$; hence, in the region where $x_{\text{first}}(t) \leq x(t) \leq a_0 t$, the velocity and density profiles have not been described. Obviously these characteristics $x(t)$ all have a $x_0 = 0$. Therefore, in this region where $x_{\text{first}}(t) \leq x \leq a_0 t$, the characteristics form a "fan," because they all start at the origin, but have different slope.

From Equation (3-10), we obtain that the characteristics in this region are

$$\frac{x}{t} = u_{\text{fan}}(x,t) + a(\rho_{\text{fan}}(x,t)). \tag{4-7}$$

Using Equation (4-2), we find that

$$u_{\text{fan}}(x,t) = \frac{2}{\gamma-1}\big(a(\rho_{\text{fan}}(x,t)) - a_0\big). \tag{4-8}$$

We can combine Equations (4-7) and (4-8) to obtain

$$u_{\text{fan}}(x,t) = \frac{2}{\gamma+1}\left(\frac{x}{t} - a_0\right), \tag{4-9}$$

$$\rho_{\text{fan}}(x,t) = \Big(\frac{2}{K(\gamma+1)}\big((\gamma-1)\frac{x}{2t} + a_0\big)\Big)^{2/(\gamma-1)}. \tag{4-10}$$

Thus we have obtained the complete solution to the problem for all $x \in \mathbb{R}$ and all $t \geq 0$. We can summarize all of the preceding results as follows:

$$u(x,t) = \begin{cases} 0, & x > a_0 t, \\ u_{\text{fan}}(x,t), & x_{\text{first}}(t) \leq x \leq a_0 t, \\ X_p'(\tau), & x < x_{\text{first}}(t); \end{cases}$$

$$\rho(x,t) = \begin{cases} \rho_0, & x > a_0 t, \\ \rho_{\text{fan}}(x,t), & x_{\text{first}}(t) \leq x \leq a_0 t, \\ \rho_{\text{piston}}(x,t), & x < x_{\text{first}}(t). \end{cases}$$

Figure 4-1 shows the space-time diagram for the complete solution above, including some of the characteristics along which the invariants travel.

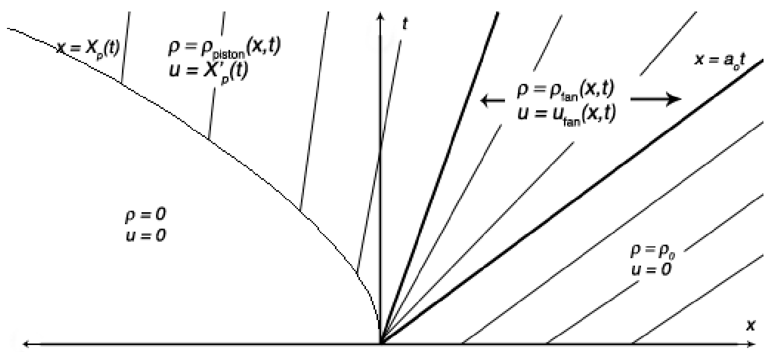

Figure 4-1. Space-time diagram for the receding piston.

Since we have made many assumptions, it is important to check the validity of the solution. In Equation (4-5), we made an implicit assumption that

$$\frac{a_0}{K} + \frac{\gamma - 1}{2K} X_p'(\tau) \geq 0;$$

otherwise, the $2/(\gamma - 1)$th power of the expression is not always real. Note that $\tau = t$ on the path of the piston, so that

$$0 \geq X_p'(t) \geq \frac{-2a_0}{\gamma - 1} \text{ for all } t \geq 0. \tag{4-11}$$

Thus the simple-wave solution exists, and is valid only for those $X_p(t)$ that satisfy the above inequality.

We have obtained the complete solution and the condition for its validity. Let us next consider the more interesting problem of an advancing piston.

5. An Advancing Piston. In this section, we consider the waves generated by the motion of an advancing piston. Once again, we assume $U_p(t) = X_p'(t)$ for some "well-behaved" $X_p(t)$; that is, $X_p(t)$ is positive and monotonically increasing for all times $t \geq 0$. Furthermore, we assume the piston is initially at the origin; that is, $X_p(0) = 0$. We appeal to the argument that no information travels toward the piston from the undisturbed area, and thus we consider only the Γ_+ characteristics. The boundary and initial conditions on this problem are the same as the ones given by Equation (4-1), except with a different $X_p(t)$ as we just assumed.

It is not hard to see that, where there were "fanlike" characteristics before, there are crossing characteristics now. Such crossings pose a serious problem since they result in a double-valued velocity and density, which is obviously unphysical. The problem can be resolved by introducing shocks. Shocks replace the crossing characteristics by introducing a jump discontinuity in the solution at the intersection of the characteristics. In other words, the solution is computed along one set of characteristics, up to the intersection of the two sets; then, the solution "jumps" to the values computed along the other set of characteristics. Then, clearly, the simple-wave solution holds on both sides of the shock. Furthermore, we must assume shocks are weak in magnitude, a reasonable assumption because $X_p(t)$ is well behaved. If the shocks are strong in magnitude, then the isentropic assumption breaks down; see [**4**, pp. 171–177] or [**3**, pp. 392–396].

A shock forms in the receding case after the expansion fan collapses beyond zero width into an inverted fan. In other words, the shock forms when the characteristics emanating from the piston intersect the characteristics emanating from the undisturbed area. Hence, the shock forms when $x_{\text{first}}(t) = a_0 t$. Using Equations (4-4) and (4-6), we conclude that the condition for shock formation is

$$a_0 + \frac{\gamma+1}{2} X_p'(0) \geq a_0, \text{ or } X_p'(0) \geq 0. \tag{5-1}$$

Consequently, a shock always forms if the piston is advancing into the gas. Notice that Equation (5-1) does not tell us when the first shock occurs, only that it does occur; Landau and Lifshitz [**3**, pp. 369–370] give the details on when the first shock occurs.

For weak shocks — that is, when the jump in u and ρ is small — the simple-wave solution can be "patched," so that it is still valid. As in the expansion-fan case, in fact, there are multiple parts to the solution; there are, in fact, two parts: one in the region ahead of the shock and one in the region behind the shock.

Whitham [**4**, p. 179] gives the proper Rankine–Hugoniot condition for the velocity of the discontinuity:

$$\frac{dx_s}{dt} = \frac{1}{2}\left(c_{\text{behind}} + c_{\text{ahead}}\right) = a_0 + \frac{\gamma+1}{4} X_p'(\tau) \tag{5-2}$$

where dx_s/dt is the shock velocity, $c = u + a$, and τ is a parameter implicitly determined by the equation. The latter implies that the propagation velocity of the shock is the average of the velocities of the Riemann invariants on either side of it; this condition is intuitive, especially since the Riemann invariants are constant in the simple-wave solution. Therefore, the position of the shock x_s is given by

$$x_s(\tau) = X_p(\tau) + \Big(a_0 + \frac{\gamma+1}{2} X_p'(\tau)\Big)(t-\tau). \tag{5-3}$$

We can solve for $x_s(t)$ by eliminating the parameter τ from Equation (5-3) and the boundary condition at the piston. Notice that Equation (5-3) is the same as Equation (4-3); they must be equal, because the shock appears at the interface where the characteristics emerging from the piston intersect the characteristics from the undisturbed area.

Hence the complete solution to the advancing piston problem is

$$u(x,t) = \begin{cases} 0, & x \geq x_s(t), \\ X'_p(\tau), & x < x_s(t); \end{cases}$$

$$\rho(x,t) = \begin{cases} \rho_0, & x \geq x_s(t), \\ \rho_{\text{piston}}(x,t), & x < x_s(t). \end{cases}$$

Figure 5-1 shows the space-time diagram for this solution, including some of the characteristics along which the invariants travel, and the position of the shock wave. Note that, in general, $x_s(t) \neq a_0 t$, contrary to the figure.

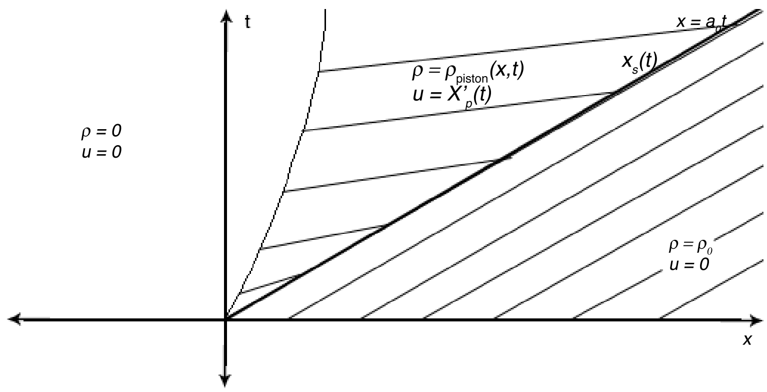

Figure 5-1. Space-time diagram for the advancing piston.

6. An Oscillating Piston. In this section, we consider an initial-value problem that exhibits both diverging characteristics and crossing characteristics, while remaining solvable analytically. Unlike the general discussions in the previous sections, we consider a piston with a constant speed α.

The piston moves into the fluid with constant speed α until $t = \eta$, then recedes with constant speed α. Hence the position of the piston is

$$X_p(t) = \begin{cases} \alpha t, & t \leq \eta; \\ \alpha(2\eta - t), & t > \eta. \end{cases} \tag{6-1}$$

As before, we can state the initial and boundary conditions as follows:

$$\begin{aligned} u(x,0) &= 0, \\ \rho(x,0) &= \rho_0, \\ u(X_p(t),t) &= X'_p(t). \end{aligned} \tag{6-2}$$

We seek the solution for all times $0 \leq t \leq 2\eta$ and the half-space $0 \leq x < \infty$. So we solve the problem for $t < \eta$ first and then for $t > \eta$, putting the two solutions together at the end to obtain the complete solution for all t.

For $t < \eta$, the piston is advancing. Therefore, we must find the position of the shock wave. So Equation (5-2) becomes

$$\frac{dx_s}{dt} = a_0 + \frac{\gamma+1}{4}\alpha,$$

and since the shock starts at the origin, we have

$$x_s(t) = \Big(a_0 + \frac{\gamma+1}{4}\alpha\Big)t.$$

So ahead of the shock where $x \geq x_s(t)$, we have $\rho = \rho_0$ and $u = 0$; however, behind the shock, we have $\rho = \rho_{\text{piston}}(x,t)$ as in Equation (4-5), and $u = \alpha$.

At $t = \eta$, the piston starts moving backwards with constant speed α. Thus as we already saw in the case of a receding piston, an expansion fan is created. Thus the solution for $t > \eta$ and $x > \alpha\eta$ is $\rho = \rho_{\text{fan}}(x - \alpha\eta, t - \eta)$ and $u = u_{\text{fan}}(x - \alpha\eta, t - \eta)$ as given by Equations (4-9) and (4-10). Finally, in the region where $x < \alpha\eta$ and $t > \eta$, we have $\rho = \rho_{\text{piston}}(x - \alpha\eta, t - \eta)$ and $u = -\alpha$.

Figure 6-1 shows the space-time diagram of the solution. The figure suggests that, in general, the fan characteristics could eventually intersect the characteristics from the undisturbed area. However, this phenomenon does not occur for the simple-wave solution.

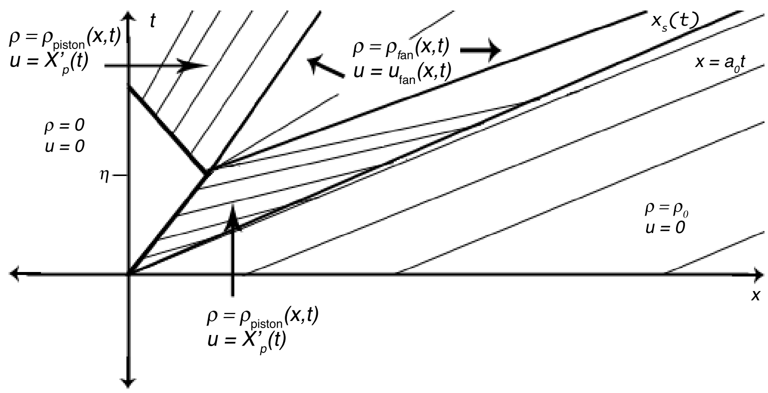

Figure 6-1. Space-time diagram of the advancing then receding oscillation.

To show analytically that these characteristics never intersect, we seek the intersection of the front edge of the shifted expansion fan with the last characteristic of initial density. Thus

$$x - \alpha\eta = a_0(t - \eta) \text{ and } x = a_0 t, \text{ or } \alpha = a_0.$$

Therefore, the characteristics intersect only if the piston velocity is equal to the speed of sound. But this velocity exceeds the maximum velocity for which the simple-wave solution is valid as given by the inequality of Equation (4-11). Hence the intersection does not occur.

However, the fan can meet the shock at some time $t^* \geq \eta$. Equating the characteristic of the leading edge of the fan with the shock position, we have

$$x - \alpha\eta = a_0(t^* - \eta) \text{ and } x = \Big(a_0 + \frac{\gamma+1}{4}\alpha\Big)t^*, \text{ or } t^* = 4\frac{a_0 - \alpha}{\gamma+1}\eta;$$

therefore, $\alpha < a_0$ since $t \geq 0$. However, we are solving only during one oscillation: that is $0 \leq t \leq 2\eta$. Thus we need to find the second shock only when

$$4\frac{a_0 - \alpha}{\gamma+1} \leq 2, \text{ or } \alpha \geq a_0 - \frac{\gamma+1}{2}.$$

If we assume α is such that the second shock does not form, then we have obtained the complete solution. Otherwise, the position of the second shock $\widetilde{x}_s(t)$ can be determined from the following ODE:

$$\frac{d\widetilde{x}_s}{dt} = \frac{1}{\gamma+1}\Big(\frac{x-\alpha\eta}{t-\eta} - a_0\Big) + \frac{a_0}{2}$$

with initial condition $\widetilde{x}_s(t^*) = x_s(t^*)$. Behind the second shock $x > \widetilde{x}_s(t)$, we have the fan as before, and ahead of the second shock $x < \widetilde{x}_s(t)$, we have the undisturbed region. Thus we have obtained the complete solution to the problem.

The symmetric problem of a piston receding then advancing can be solved in the same manner. However, the expansion fan is created first, then it is compressed. A piecewise shock wave forms. We do not go into the details of this case.

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