Solitary Internal Waves in the Ocean

The Scattering Transform Perspective

Ivan Christov*

Code 7181, Naval Research Laboratory,
Stennis Space Center, MS 39529, USA

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Background & Introduction (I)

- **1895**: Korteweg and de Vries derived the simplest known equation, which models both nonlinearity and dispersion, as the leading order approximation of the eq'ns of fluid dynamics for long waves in shallow water.
- **1965**: Zabusky and Kruskal conducted numerical experiments of the continuum limit of the Fermi-Pasta-Ulam lattice (= the KdV equation!); discovered *soliton interactions*.
- **1967**: Greene *et al.* showed the equivalence of the initial value problems for the KdV on the infinite line to the classical Schrödinger eigenvalue problem.
- **1973**: Extending the latter, Ablowitz *et al.* (AKNS) formulated a general method for solving nonlinear evolution equations known as the *Scattering Transform*.

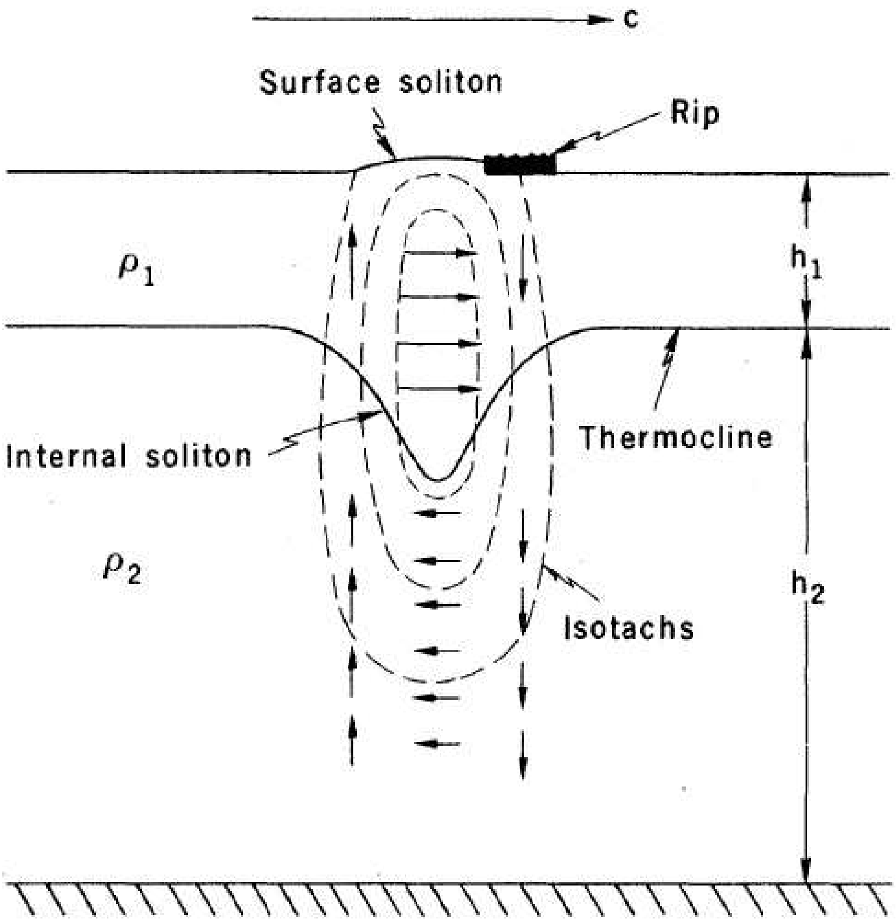
Background & Introduction (II)

- **1975-6**: Matveev *et al.*, concurrently with Flaschka and McLaughlin, used methods from algebraic geometry to develop the theory of the *periodic* Scattering Transform; a crucial advance for the future application of the S.T. to data analysis.
- **1980s–1990s**: Osborne applied and extended the techniques of the Scattering Transform to the analysis of oceanographic data; developed the first (and only) practical approach to the *nonlinear Fourier analysis* of physical data.
- **Present day**: Apply Osborne's nonlinear Fourier analysis framework to data set of internal solitons in the ocean, since (linear) Fourier analysis has proven ineffective when dealing with *nonlinear* phenomena.

Position of the Problem

KdV: $\eta_t + c_0\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = 0;\ 0 \le x \le L,\ t \ge 0;$

IBC: $\eta(x,0) = \eta_0(x),\ \eta(x+L,0) = \eta(x,0).$



The physical parameters are

$$c_0 \simeq \sqrt{g\frac{\Delta\rho}{\rho}\left(\frac{h_1h_2}{h_1+h_2}\right)},$$

$$\alpha \simeq \frac{-3c_0}{2}\left(\frac{h_2-h_1}{h_1h_2}\right),$$

$$\beta \simeq \frac{1}{6}c_0h_1h_2,$$

$$\rho \simeq \rho_1 \simeq \rho_2.$$

Fourier Analysis of Linear PDEs

Consider the linearized KdV equation (i.e. $\alpha = 0$):

$$\eta_t + c_0\eta_x + \beta\eta_{xxx} = 0.$$

We can find its *exact* solution in terms of a Fourier Series:

1. **“Discrete Fourier Transform” (DFT)**: Find the spectrum $\{c_j; \phi_j\}$: $c_j = \sqrt{a_j^2 + b_j^2}$, $\phi_j = \tan^{-1}\left(\frac{-b_j}{a_j}\right)$, where a_j and b_j are the usual Fourier coefficients.
2. **“Inverse Discrete Fourier Transform” (IDFT)**: Construct the solution to the linear PDE from the spectrum $\{c_j; \phi_j\}$:

$$\eta(x,t) = \frac{a_0}{2} + \sum_{j=1}^{N} c_j \cos(k_j x - \omega_j t + \phi_j),$$

where $\omega_j = c_0 k_j - \beta k_j^3$ from the dispersion relation.

Integrable Nonlinear PDEs

Consider the (nonlinear) KdV equation:

$$\eta_t + c_0\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = 0.$$

We can find its *exact* solution via the Scattering Transform:

1. **Direct Scattering Transform (DST)**: Solve the Schrödinger eigenvalue problem:

   $$\Psi_{xx} + [\lambda\eta(x,0) + E]\Psi = 0,$$

   where $\lambda = \frac{\alpha}{6\beta}$ (the nonlinearity to dispersion ratio), to get the discrete spectrum, aka "scattering data," $\{E_j; \mu_j\}$.

2. **Inverse Scattering Transform (IST)**: Construct the *nonlinear* Fourier series from the spectrum $\{E_j; \mu_j\}$ using hyperelliptic functions of the Riemann Θ-function.

The Nonlinear Fourier Series

- In terms of the hyperelliptic (aka Abelian) functions:

$$\lambda\eta(x,t) = -E_1 + \sum_{j=1}^{N} 2\mu_j(x,t) - E_{2j} - E_{2j+1}.$$

  - All *nonlinear* waves and their interactions are obtained from this *linear* superposition.
- In the small amplitude limit, i.e. $|\mu_j(x,t)| \ll 1$, we have $\mu_j(x,t) \to \cos(x - \omega_j t + \phi_j)$, i.e. we get the ordinary Fourier series!
- If there are no interactions, e.g. a single wave ($N = 1$), we have $\mu(x,t) \to \mathrm{cn}^2(x - \omega t + \phi|m)$, which is a Jacobian elliptic function with modulus m (a *cnoidal* wave).

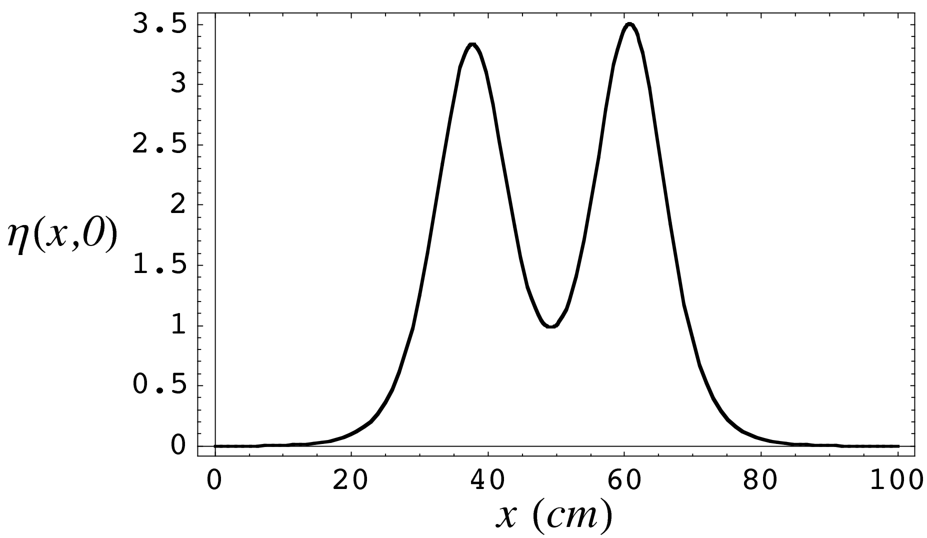
On Moduli and Wave Numbers

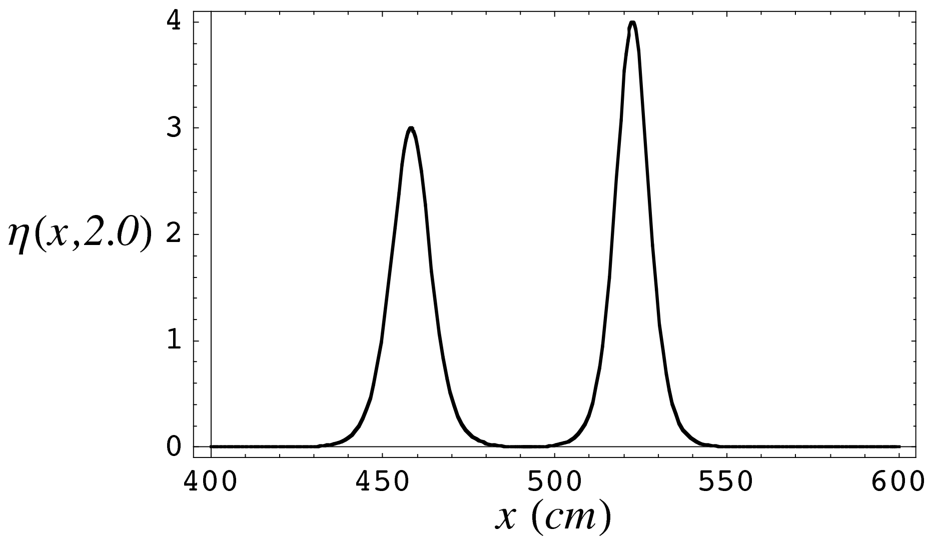
- In the hyperelliptic representation, we have $k_j = 2\pi j/L$ as in Fourier theory ("commensurable" wave numbers).
- Furthermore, we can compute the modulus m_j of the hyperelliptic functions, which we call the "*soliton index*," from the discrete spectrum as

$$m_j = \frac{E_{2j+1} - E_{2j}}{E_{2j+1} - E_{2j-1}}, \quad 1 \leq j \leq N.$$

  Then, the oscillations fall into three general categories:
  - $m_j \gtrsim 0.99 \Rightarrow$ solitons, i.e. $\mathrm{cn}^2(x|m_j = 1) = \mathrm{sech}^2(x)$,
  - $m_j \gtrsim 0.5 \Rightarrow$ nonlinearly interacting cnoidal waves, "less nonlinear" than the solitons,
  - $m_j \ll 1.0 \Rightarrow$ radiation, i.e. $\mathrm{cn}^2(x|m_j = 0) = \cos^2(x)$.

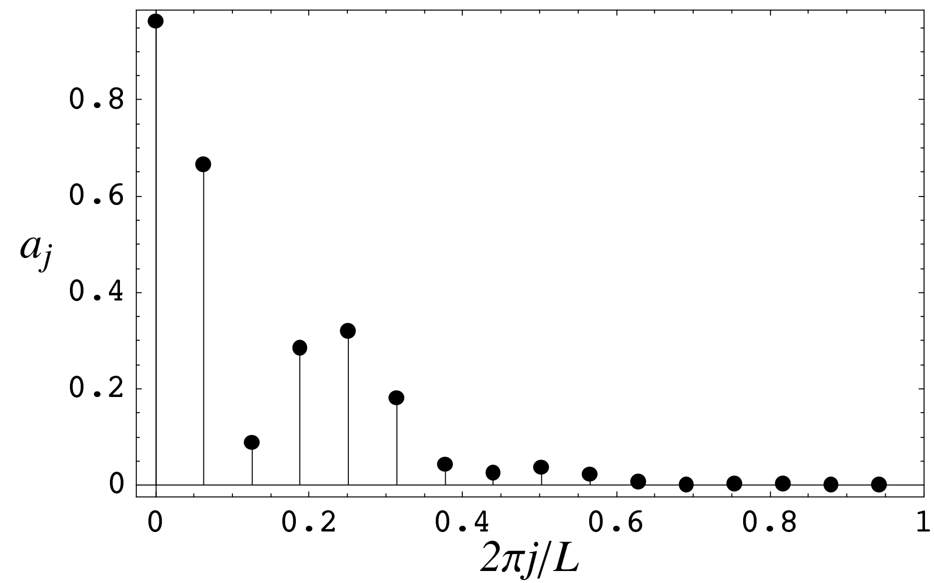
A Two-Soliton Initial Condition

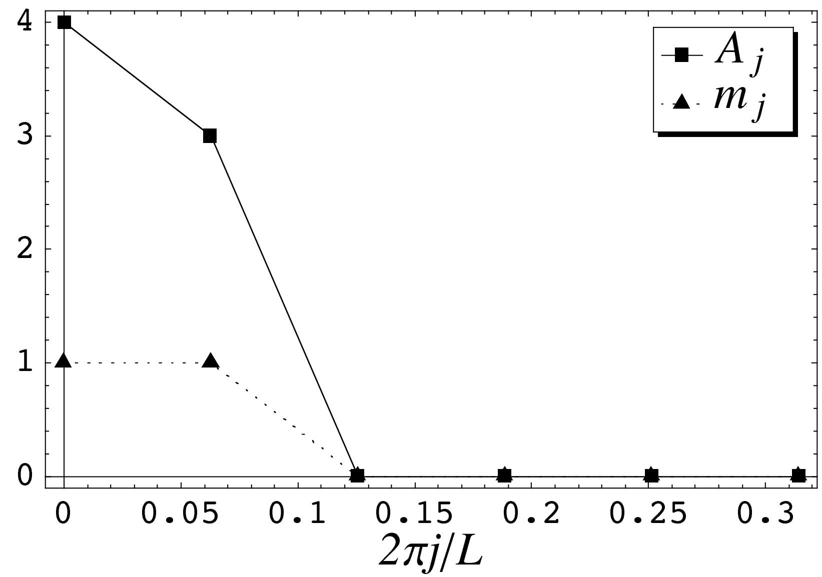




- (Left) $\eta(x+100,0) = \eta(x,0)$ is the $N = 2$ case of the well-known infinite line N-soliton solution (i.e. Hirota's construction), taken at $t = 0.15$s for convenince.
- (Right) The same 2-soliton solution but at $t = 2.0$s, note the amplitudes of the two solitons are 3.0cm and 4.0cm, respectively.

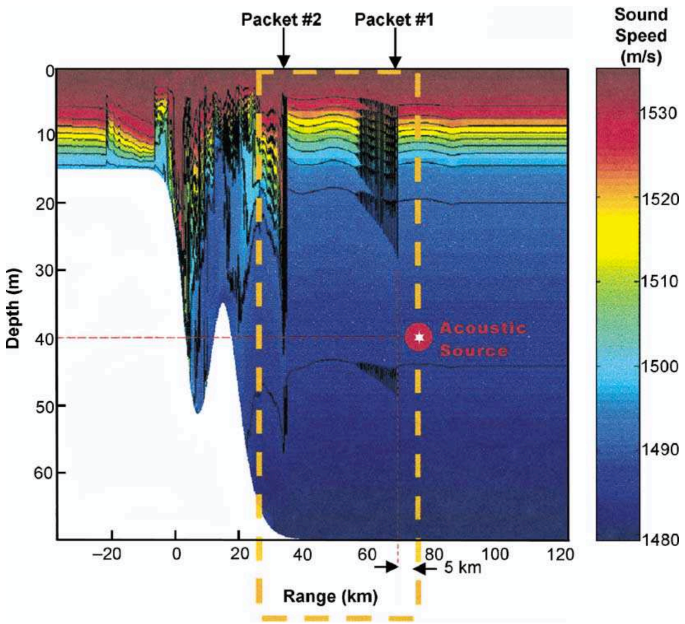
Comparison of the FFT & DST (I)

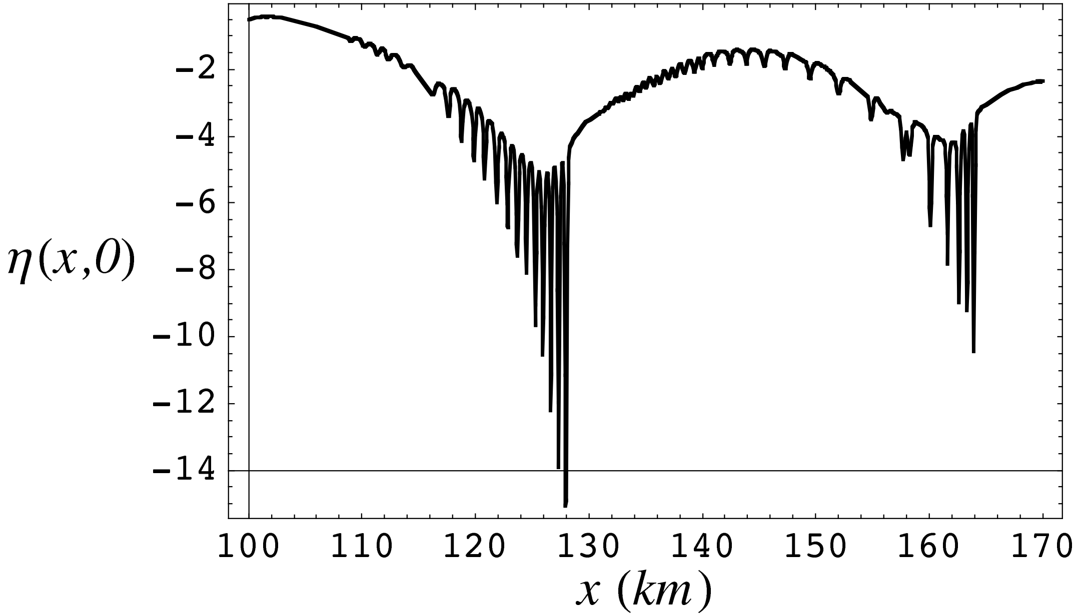




- (Left) The DFT finds ≈ 16 normal modes.
- (Right) The DST finds *exactly* 2 significant waves.
- **N.B.**: When the data, i.e. the i.c., is a solution of the KdV eq. (even $\approx$), the DST/IST offer a far better interpretation/representation of the data than do the DFT/IDFT.

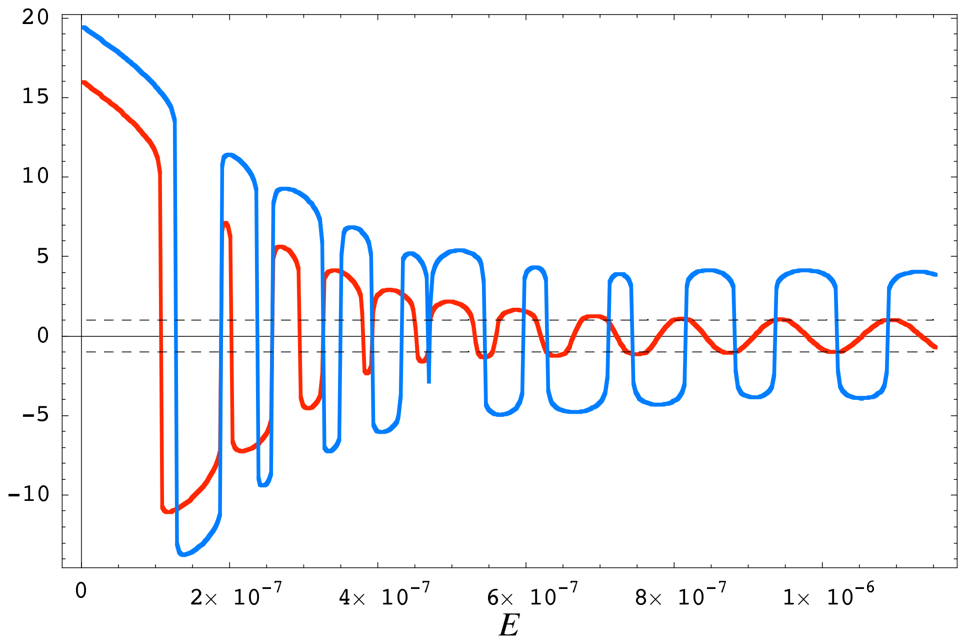
Internal Solitons in the Yellow Sea





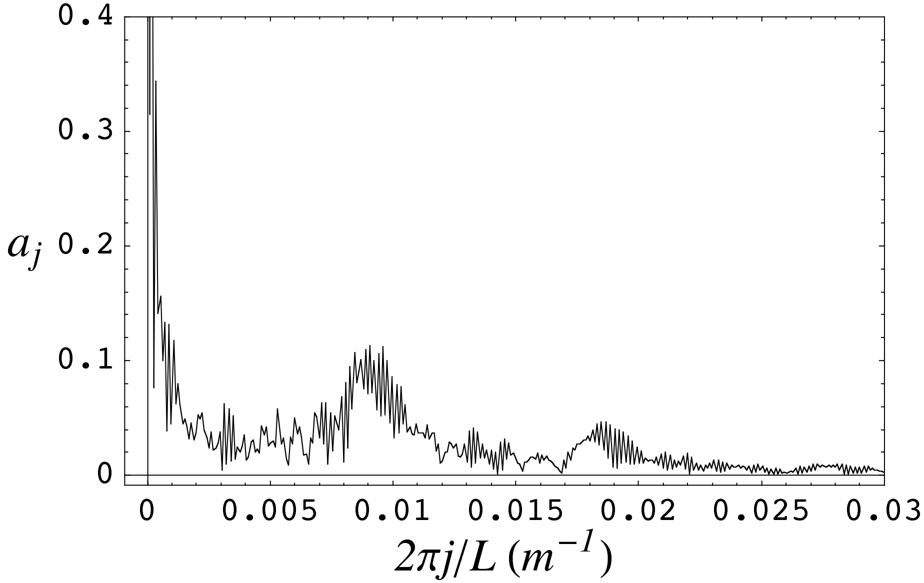
- (Left) The profile on internal waves in the Yellow Sea, generated via numerical simulations, using the nonhydrostatic Lamb model, by Chin-Bing *et al.*.
- (Right) The wave packets we are interested in; part of the $\sigma_t = 22.0$ isopycnal, which is at a depth of ≈ 58m.

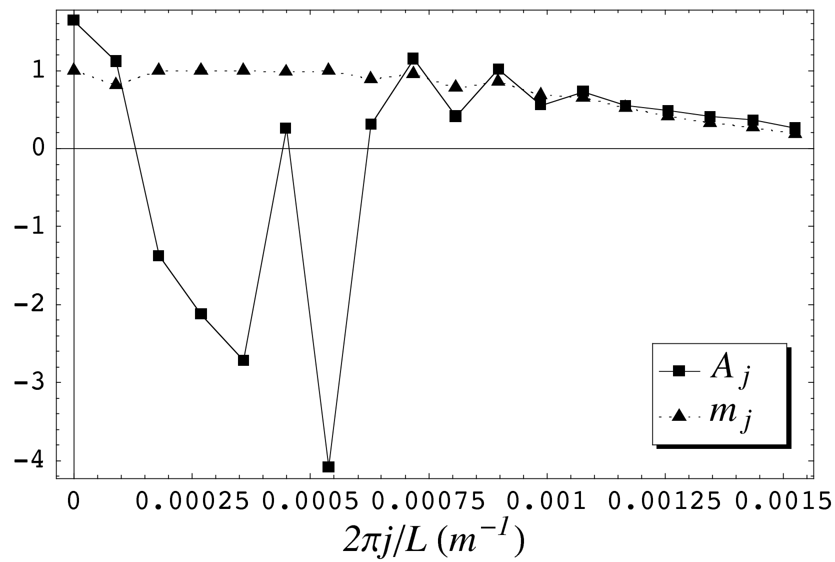
About the Numerics of the DST



- The zero-crossings of the blue oscillations determine the number of the "degrees of freedom" in the DST spectrum; each crossing is μ_j value.
- Consecutive $+1$ or -1 crossings of the red oscillations determine the "open bands" of the DST spectrum; each crossing is an E_k value.

Comparison of the FFT & DST (II)





- Above: on left is the DFT and on the right the DST.
- Three things are to be noted about the DST spectrum:
  - the *narrower* (and distinct) range of wave numbers,
  - the *nonlinear non-soliton* waves in the spectrum,
  - there are fewer oscillations modes predicted.

Acknowledgments, etc.

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  - Dr. Chin-Bing for suggesting this research project and giving me the opportunity to work on it at NRL-SSC this summer,
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- I will begin my graduate studies in mathematics at Texas A&M University next week. For correspondence purposes, my email there is `christov@tamu.edu`.
- The algorithms for the Direct Scattering Transform were implemented in Mathematica v. 5.1 and all the graphics generated with the latter.

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