Nonlinear Fourier Analysis

The Direct & Inverse Scattering Transforms for the Korteweg–de Vries Equation

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Background & Introduction (I)

- **1895**: Korteweg and de Vries derived the simplest known equation that models both nonlinearity and dispersion, as the leading order approximation of the eq'ns of fluid dynamics, for long waves in shallow-water.
- **1965**: Zabusky and Kruskal conducted numerical experiments of continuum limit of the Fermi-Pasta-Ulam lattice (= the KdV equation!); discovered *soliton interactions*.
- **1967**: Greene *et al.* showed the equivalence of the initial value problems for the KdV on the infinite line to the classical Schrödinger eigenvalue problem.
- **1973**: Extending the latter, Ablowitz *et al.* (AKNS) formulated a general method for solving nonlinear evolution equations known as the *Scattering Transform*.

Background & Introduction (II)

- **1975-6**: Matveev *et al.*, concurrently with Flaschka and McLaughlin, used methods from algebraic geometry to develop the theory of the *periodic* Scattering Transform; crucial work for the application of the S.T. to data analysis.
- **1980s–1990s**: Osborne extended the technique of the Scattering Transform to the analysis of oceanographic data; develops the first (and only) practical approach to the *nonlinear Fourier analysis* of physical data.
- **Present day**: Apply Osborne's nonlinear Fourier analysis framework to the analysis of internal solitons in the ocean, since (linear) Fourier analysis has proven ineffective for the analysis of *nonlinear* phenomena.

Fourier Analysis of Linear PDEs

Consider the linearized KdV equation with periodic IC:

$$\eta_t + c_0\eta_x + \beta\eta_{xxx} = 0,\ \eta(x+L,0) = \eta(x,0),\ 0 \le x \le L,$$

where $c_0 = \sqrt{gh}$, $\beta = c_0 h^2/6$. The *exact* solution to the preceding is a Fourier Series:

1. **"Discrete Fourier Transform" (DFT)**: Find the spectrum $\{c_j; \phi_j\}$: $c_j = \sqrt{a_j^2 + b_j^2}$, $\phi_j = \tan^{-1}\left(\frac{-b_j}{a_j}\right)$, where a_j and b_j are the usual Fourier coefficients.
2. **"Inverse Discrete Fourier Transform" (IDFT)**: Construct the solution to the linear PDE from the spectrum $\{c_j; \phi_j\}$:

$$\eta(x,t) = \frac{a_0}{2} + \sum_{j=1}^{N} c_j \cos(k_j x - \omega_j t + \phi_j),\ \omega_j = c_0 k_j - \beta k_j^3.$$

Integrable Nonlinear PDEs

Consider the full KdV equation with periodic IC:

$$\eta_t + c_0\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = 0,\ \eta(x+L,0) = \eta(x,0),\ 0 \le x \le L,$$

$\alpha = 3c_0/(2h)$. We can find the *exact* solution to the preceding via the Scattering Transform:

1. **Direct Scattering Transform (DST)**: Solve the Schrödinger eigenvalue problem

$$\Psi_{xx} + [\lambda\eta(x,0) + E]\Psi = 0,$$

   where $\lambda = \alpha/(6\beta)$, to get the discrete spectrum, aka "scattering data," $\{E_j; \mu_j\}$.

2. **Inverse Scattering Transform (IST)**: Construct the *nonlinear* Fourier series from the spectrum $\{E_j; \mu_j\}$.

The Nonlinear Fourier Series

- In terms of the hyperelliptic (aka Abelian) functions:

$$\lambda\eta(x,t) = -E_1 + \sum_{j=1}^{N} 2\mu_j(x,t) - E_{2j} - E_{2j+1}.$$

  - All *nonlinear* waves and their interactions are obtained from this *linear* superposition.
- In the small amplitude limit we have $\mu_j(x,t) \to \cos(x - \omega_j t + \phi_j)$, i.e., we get the ordinary Fourier series!
- If there are no interactions, e.g., a single wave ($N = 1$), we have $\mu(x,t) \to \mathrm{cn}^2(x - \omega t + \phi|m)$, which is a Jacobian elliptic function with modulus m (a *cnoidal* wave).

The Θ-function Representation

- In terms of the N-degree-of-freedom Riemann Θ-function:

$$\eta(x,t) = \frac{2}{\lambda}\frac{\partial^2}{\partial x^2}\ln\Theta_N(\boldsymbol{\eta}|\mathbf{B}) = \eta_{\text{cn}} + \eta_{\text{int}},$$

where $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_N)$, $\eta_j = k_j x - \omega_j t + \phi_j$ and $\mathbf{B}$ is the "interaction matrix."

  - The solution is a *linear* superposition of *cnoidal* waves, and their *nonlinear* interactions.

More on the Θ-function Repr'n

The *linear* superposition of cnoidal waves is given by

$$\eta_{\text{cn}} = \frac{2}{\lambda}\frac{\partial^2}{\partial x^2}\ln\Theta_N(\boldsymbol{\eta}|\mathbf{D}) = 2\sum_{j=1}^{N} A_j\,\mathrm{cn}^2\left[\frac{K(m_j)}{\pi}\cdot k_j(x - C_n t)\middle| m_j\right].$$

But, their *nonlinear interactions* are still in terms of Θ-functions:

$$\eta_{\text{int}} = \frac{2}{\lambda}\frac{\partial^2}{\partial x^2}\ln\left[1 + \frac{\Theta_N(\boldsymbol{\eta},\mathbf{B}) - \Theta_N(\boldsymbol{\eta}|\mathbf{D})}{\Theta_N(\boldsymbol{\eta}|\mathbf{D})}\right].$$

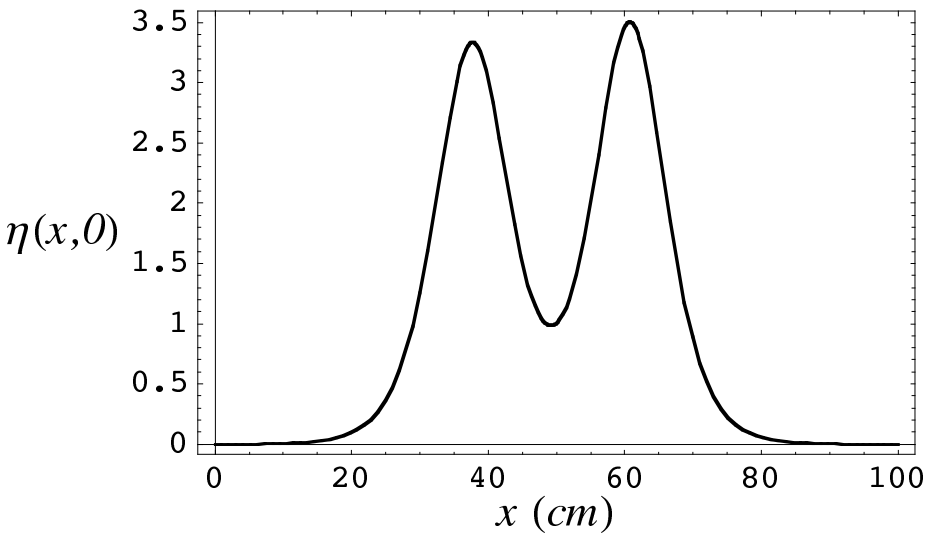
Note that $\mathbf{B} = \mathbf{D} + \mathbf{O}$, i.e., we've split the interaction matrix into its diagonal and off-diagonal parts.

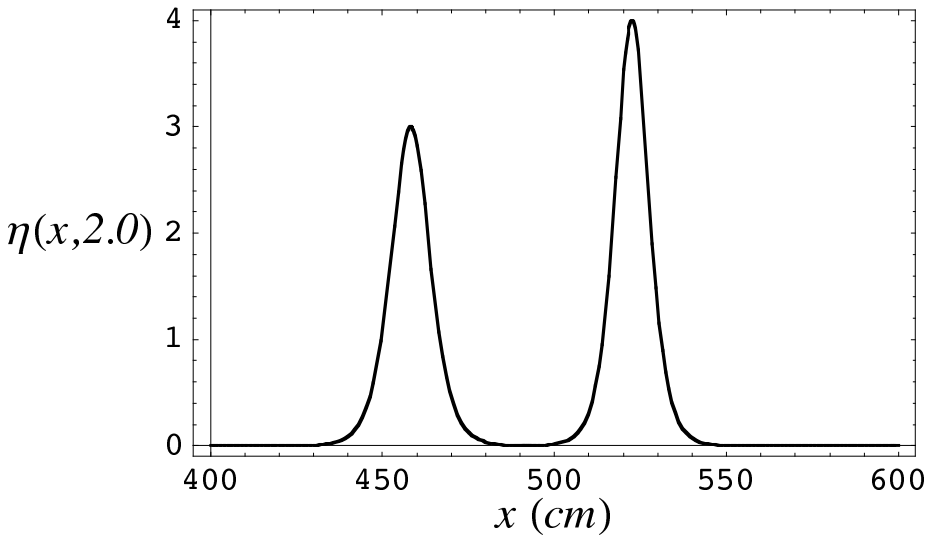
Bad news: in general, $\eta_{\text{int}} \sim \mathcal{O}(1)$, so we **cannot** ignore it!

On Moduli and Wave Numbers

- In the hyperelliptic representation, we can let $k = 2\pi j/L$ as usual ("commensurable" wave numbers).
- In the Θ-function representation the values of k_j, ω_j, ϕ_j are forced by the "Abelian transformation." Though they are analogous to the Fourier ones, they are *physically irrelevant* ("incommensurable" wave numbers).
- Furthermore, we can compute the modulus m_j of the Jacobian elliptic functions, which we call the "soliton index" from the discrete spectrum. Then,
  - $m_j \gtrsim 0.99 \Rightarrow$ solitons, i.e., $\mathrm{cn}^2(x|m_j = 1) = \mathrm{sech}^2(x)$,
  - $m_j \gtrsim 0.5 \Rightarrow$ Stokes waves (nonlinearly interacting cnoidal waves, "less nonlinear" than the solitons),
  - $m_j \ll 1.0 \Rightarrow$ radiation, i.e., $\mathrm{cn}^2(x|m_j = 0) = \cos^2(x)$.

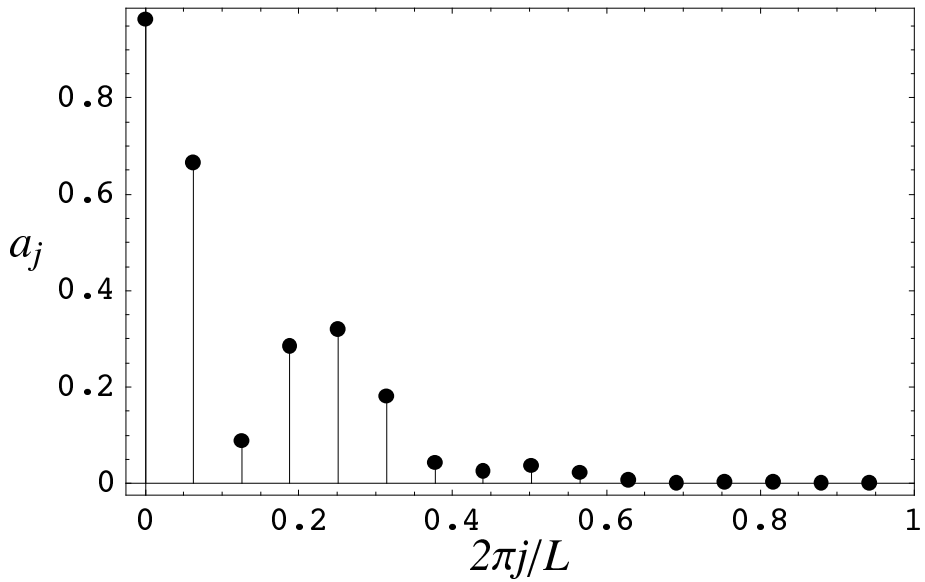
A Two-Soliton Initial Condition

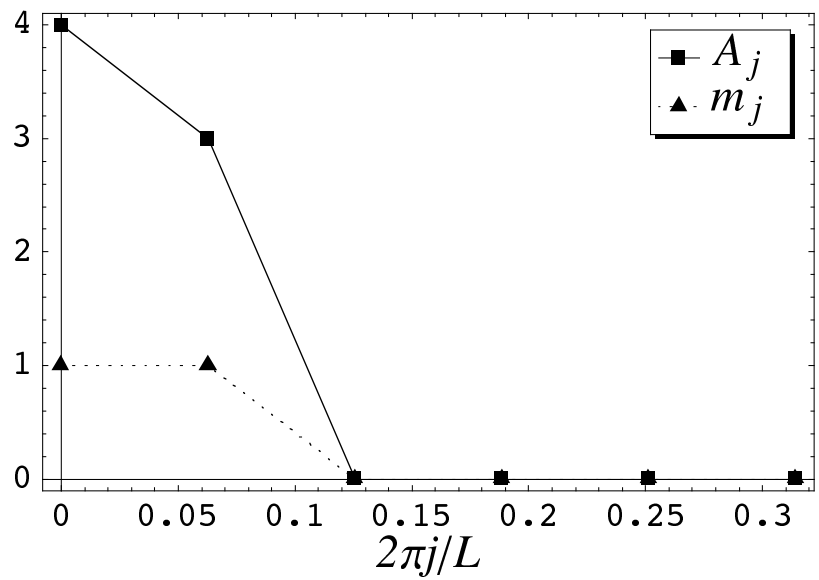




- (Left) $\eta(x+100,0)=\eta(x,0)$ is the $N=2$ case of the well-known infinite line N-soliton solution (i.e., Hirota's construction), taken at $t=0.15$s for convenince.
- (Right) The same 2-soliton solution but at $t=2.0$s, note the amplitudes of the two solitons are 3.0cm and 4.0cm, respectively.

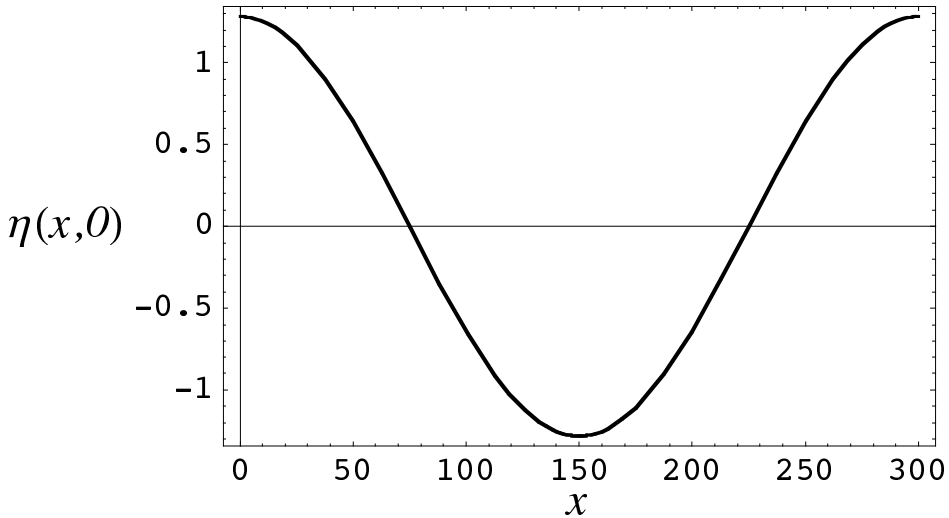
Comparison of the FFT & DST (I)

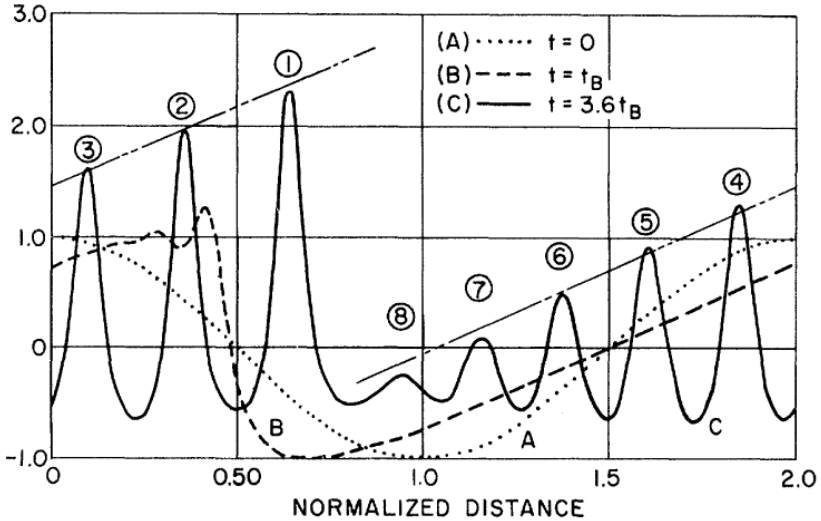


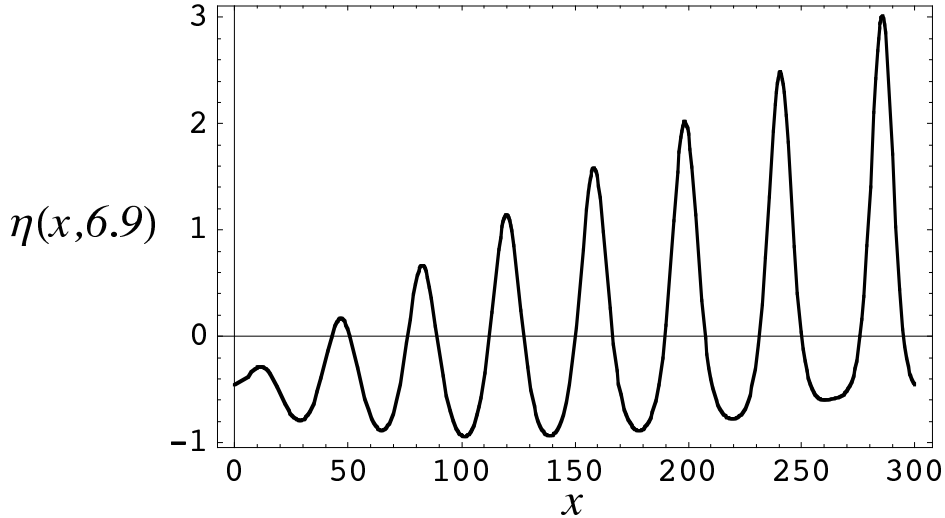


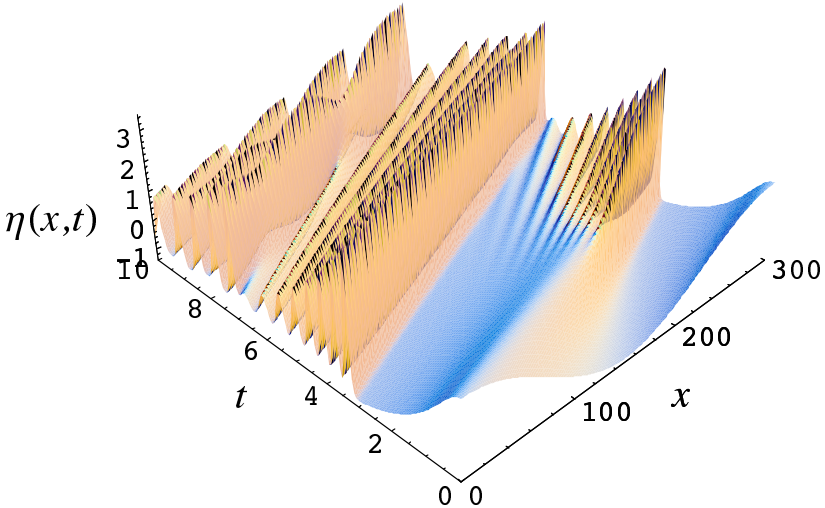
- (Left) The DFT finds ≈ 16 normal modes.
- (Right) The DST finds *exactly* 2 significant waves.
- **N.B.**: When the data, i.e., the i.c., is a solution of the KdV eq. (even $\approx$), the DST/IST offer a far better interpretation/representation of the data than do the DFT/IDFT.

A Sine Wave I.C. for the KdV Eq.



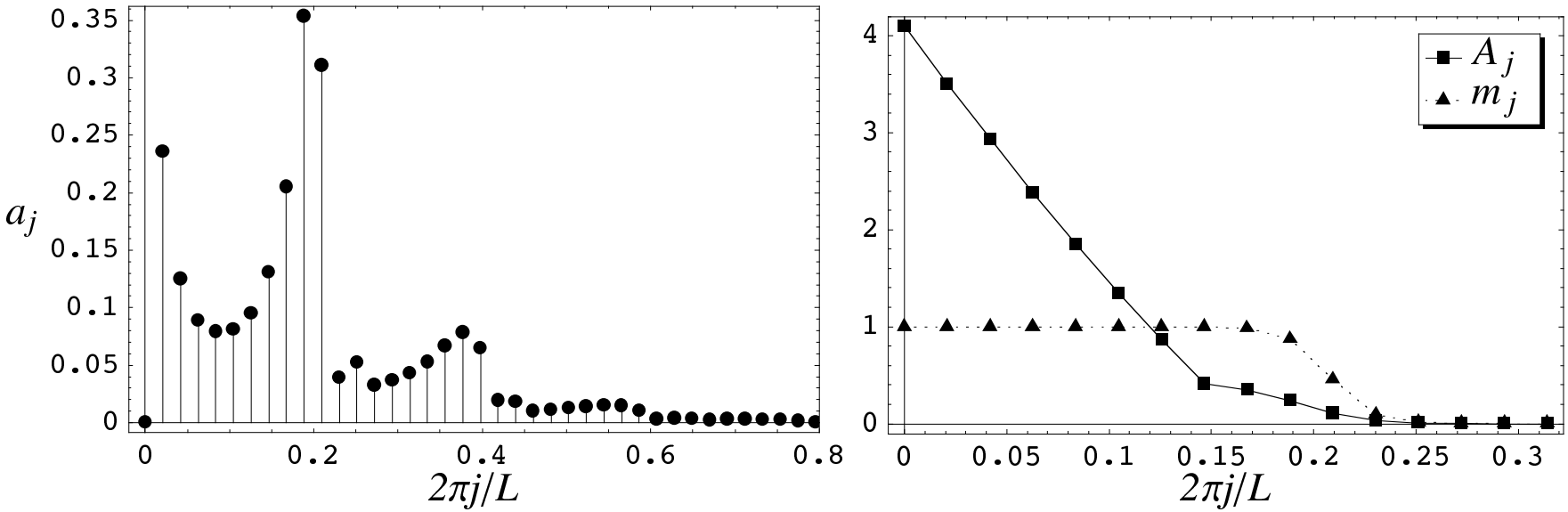






I.C.: $\eta(x+300,0) = \eta(x,0) = 1.281\cos(2\pi x/300)$

Comparison of the FFT & DST (II)



- (Left) The DFT spectrum shows 45+ normal modes.
- (Right) The DST finds 8 (well-defined) soliton modes (=Z&K), some mildly nonlinear waves, and radiation.
- **N.B.**: The FT is only useful when analyzing a *solution*. But, the DST gives all the info about an *initial condition* and its *evolution*!

Acknowledgments, etc.

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- I will begin my graduate studies in mathematics at Texas A&M University next week. For correspondence purposes, my email there is `christov@tamu.edu`.

Selected Bibliography

1. Osborne, A. R. (1998) *Physica D* **123**.
2. Osborne, A. R. (1993) *Phys. Rev. E* **48** (1).
3. Osborne, A. R. (1995) *Phys. Rev. E* **52** (1).
4. Osborne, A. R. & Petti, M. (1994) *Physics of Fluids* **6** (5).
5. Osborne, A. R. & Bergamasco, L. (1986) *Physica* **18D**.
6. Osborne, A. R. (1994) *Math. Comp. Sim.* **36**.
7. Osborne, A. R. & Burch, T. L. (1980) *Science* **208** (4443).
8. Zabusky, N. J. & Kruskal, M. D. (1965) *Phys. Rev. Lett.* **15** (6).
9. Gardner, C., Greene, J., Kruskal, M. & Miura, R. (1967) *Phys. Rev. Lett.* **19** (19).
10. Ablowitz, M. J., Kaup, D. J., Newell, A. C. & Segur, H. (1973) *Phys. Rev. Lett.* **31** (2).
11. Korteweg, D. J. & de Vries, G. (1895) *Phil. Mag.* **39**.
12. Ablowitz, M. J. & Segur, H., (1981) *Solitons and the Inverse Scattering Transform* (Philadelphia: SIAM Studies in Applied Mathematics).