Nonlinear Fourier Analysis

The Direct & Inverse Scattering Transforms for the Korteweg–de Vries Equation

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1895: Korteweg and de Vries derived the simplest known equation that models both nonlinearity and dispersion, as the leading order approximation of the eq’ns of fluid dynamics, for long waves in shallow-water.

1965: Zabusky and Kruskal conducted numerical experiments of continuum limit of the Fermi-Pasta-Ulam lattice (= the KdV equation!); discovered soliton interactions.

1967: Greene et al. showed the equivalence of the initial value problems for the KdV on the infinite line to the classical Schrödinger eigenvalue problem.

1973: Extending the latter, Ablowitz et al. (AKNS) formulated a general method for solving nonlinear evolution equations known as the Scattering Transform.
1975-6: Matveev *et al.*, concurrently with Flaschka and McLaughlin, used methods from algebraic geometry to develop the theory of the *periodic* Scattering Transform; crucial work for the application of the S.T. to data analysis.

1980s–1990s: Osborne extended the technique of the Scattering Transform to the analysis of oceanographic data; develops the first (and only) practical approach to the *nonlinear Fourier analysis* of physical data.

Present day: Apply Osborne’s nonlinear Fourier analysis framework to the analysis of internal solitons in the ocean, since (linear) Fourier analysis has proven ineffective for the analysis of *nonlinear* phenomena.
Consider the linearized KdV equation with periodic IC:

\[ \eta_t + c_0 \eta_x + \beta \eta_{xxx} = 0, \quad \eta(x + L, 0) = \eta(x, 0), \quad 0 \leq x \leq L, \]

where \( c_0 = \sqrt{gh} \), \( \beta = c_0 h^2 / 6 \). The exact solution to the preceding is a Fourier Series:

1. "Discrete Fourier Transform" (DFT): Find the spectrum \( \{c_j; \phi_j\} \):
   \[ c_j = \sqrt{a_j^2 + b_j^2}, \quad \phi_j = \tan^{-1}\left( \frac{-b_j}{a_j} \right), \]
   where \( a_j \) and \( b_j \) are the usual Fourier coefficients.

2. "Inverse Discrete Fourier Transform" (IDFT): Construct the solution to the linear PDE from the spectrum \( \{c_j; \phi_j\} \):
   \[ \eta(x, t) = \frac{a_0}{2} + \sum_{j=1}^{N} c_j \cos(k_j x - \omega_j t + \phi_j), \quad \omega_j = c_0 k_j - \beta k_j^3. \]
Consider the full KdV equation with periodic IC:

\[ \eta_t + c_0 \eta_x + \alpha \eta \eta_x + \beta \eta_{xxx} = 0, \quad \eta(x + L, 0) = \eta(x, 0), \quad 0 \leq x \leq L, \]

\[ \alpha = 3c_0/(2h). \] We can find the exact solution to the preceding via the Scattering Transform:

1. **Direct Scattering Transform (DST):** Solve the Schrödinger eigenvalue problem

   \[ \Psi_{xx} + \left[ \lambda \eta(x, 0) + E \right] \Psi = 0, \]

   where \( \lambda = \alpha/(6\beta) \), to get the discrete spectrum, aka “scattering data,” \( \{E_j; \mu_j\} \).

2. **Inverse Scattering Transform (IST):** Construct the nonlinear Fourier series from the spectrum \( \{E_j; \mu_j\} \).
The Nonlinear Fourier Series

- In terms of the hyperelliptic (aka Abelian) functions:

\[ \lambda \eta(x, t) = -E_1 + \sum_{j=1}^{N} 2\mu_j(x, t) - E_{2j} - E_{2j+1}. \]

- All nonlinear waves and their interactions are obtained from this linear superposition.

- In the small amplitude limit we have

  \[ \mu_j(x, t) \rightarrow \cos(x - \omega_j t + \phi_j), \text{ i.e., we get the ordinary Fourier series!} \]

- If there are no interactions, e.g., a single wave \((N = 1)\), we have

  \[ \mu(x, t) \rightarrow \cn^2(x - \omega t + \phi|m), \text{ which is a Jacobian elliptic function with modulus } m \text{ (a cnoidal wave).} \]
The $\Theta$-function Representation

- In terms of the $N$-degree-of-freedom Riemann $\Theta$-function:

$$
\eta(x, t) = \frac{2}{\lambda} \frac{\partial^2}{\partial x^2} \ln \Theta_N(\eta|B) = \eta_{\text{cn}} + \eta_{\text{int}},
$$

where $\eta = (\eta_1, \ldots, \eta_N)$, $\eta_j = k_j x - \omega_j t + \phi_j$ and $B$ is the “interaction matrix.”

- The solution is a linear superposition of cnoidal waves, and their nonlinear interactions.
More on the $\Theta$-function Repr’n

The linear superposition of cnoidal waves is given by

$$\eta_{cn} = \frac{2}{\lambda} \frac{\partial^2}{\partial x^2} \ln \Theta_N(\eta|\mathbf{D}) = 2 \sum_{j=1}^{N} A_j \ cn^2 \left[ \frac{K(m_j)}{\pi} \cdot k_j(x - C_n t) \right] m_j.$$

But, their nonlinear interactions are still in terms of $\Theta$-functions:

$$\eta_{int} = \frac{2}{\lambda} \frac{\partial^2}{\partial x^2} \ln \left[ 1 + \frac{\Theta_N(\eta, \mathbf{B}) - \Theta_N(\eta|\mathbf{D})}{\Theta_N(\eta|\mathbf{D})} \right].$$

Note that $\mathbf{B} = \mathbf{D} + \mathbf{O}$, i.e., we’ve split the interaction matrix into its diagonal and off-diagonal parts.

Bad news: in general, $\eta_{int} \sim \mathcal{O}(1)$, so we cannot ignore it!
On Moduli and Wave Numbers

- In the hyperelliptic representation, we can let $k = 2\pi j / L$ as usual ("commensurable" wave numbers).

- In the $\Theta$-function representation the values of $k_j, \omega_j, \phi_j$ are forced by the "Abelian transformation." Though they are analogous to the Fourier ones, they are physically irrelevant ("incommensurable" wave numbers).

- Furthermore, we can compute the modulus $m_j$ of the Jacobian elliptic functions, which we call the "soliton index" from the discrete spectrum. Then,
  - $m_j \gtrsim 0.99 \Rightarrow$ solitons, i.e., $\text{cn}^2(x|m_j = 1) = \text{sech}^2(x)$,
  - $m_j \gtrsim 0.5 \Rightarrow$ Stokes waves (nonlinearly interacting cnoidal waves, "less nonlinear" than the solitons),
  - $m_j \ll 1.0 \Rightarrow$ radiation, i.e., $\text{cn}^2(x|m_j = 0) = \cos^2(x)$. 

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(Left) \( \eta(x + 100, 0) = \eta(x, 0) \) is the \( N = 2 \) case of the well-known infinite line \( N \)-soliton solution (i.e., Hirota’s construction), taken at \( t = 0.15s \) for convenience.

(Right) The same 2-soliton solution but at \( t = 2.0s \), note the amplitudes of the two solitons are 3.0\( \text{cm} \) and 4.0\( \text{cm} \), respectively.
(Left) The DFT finds $\approx 16$ normal modes.

(Right) The DST finds exactly 2 significant waves.

N.B.: When the data, i.e., the i.c., is a solution of the KdV eq. (even $\approx$), the DST/IST offer a far better interpretation/representation of the data than do the DFT/IDFT.
A Sine Wave I.C. for the KdV Eq.

I.C.: \( \eta(x + 300, 0) = \eta(x, 0) = 1.281 \cos\left(\frac{2\pi x}{300}\right) \)
Comparison of the FFT & DST (II)

(Left) The DFT spectrum shows 45+ normal modes.

(Right) The DST finds 8 (well-defined) soliton modes (=Z&K), some mildly nonlinear waves, and radiation.

N.B.: The FT is only useful when analyzing a solution. But, the DST gives all the info about an initial condition and its evolution!
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