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Motivation:

- Why unstructured triangulations when (Cartesian) tensor product grids are so simple and efficient to implement?
- How should we design “genuinely multidimensional” schemes and limiters on unstructured grids?

Background:

- On October 30 there were 18,630 articles on MathSciNet with primary or secondary MSC matching 35L* (Partial differential equations of hyperbolic type).
- Extension to unstructured grids: Arminjon et al. (1997).
A Picture is Worth a Thousand Words

Source: http://www.fluidmech.net/gallery/ss_cars/sscars.htm
Outline of the Talk

1. Intro to hyperbolic systems of conservation laws.
2. Crash course on Godunov-type central schemes.
   - Derivation of a fully-discrete second-order-accurate scheme.
     - 1D Example: the Nessyahu–Tadmor scheme.
     - 2D Example: our new scheme on unstructured triangulations.
   - The minimum-angle-plane reconstruction.
   - How to construct the dual (or staggered mesh) of an unstructured triangulation.
   - Scalar equation with convex flux: inviscid Burgers.
   - System of equation with convex flux: Euler equations of gas dynamics.
   - Scalar equation with nonconvex flux.
Consider the initial-boundary-value problem for a \((d + 1)\)D hyperbolic system of conservation laws:

\[
\begin{align*}
\tilde{q}_t + \nabla \tilde{x} \cdot \tilde{F}(\tilde{q}) &= \tilde{s}(\tilde{q}, \tilde{x}, t), & (\tilde{x}, t) &\in \Omega \times (0, T], \\
\tilde{q}(\tilde{x}, t = 0) &= \tilde{q}_0(\tilde{x}), & \tilde{x} &\in \Omega, \\
\tilde{q}(\tilde{x}, t) &= \tilde{q}_{BC}(\tilde{x}, t), & (\tilde{x}, t) &\in \partial \Omega \times (0, T].
\end{align*}
\]

- \(\Omega \subset \mathbb{R}^d\) is the interior of a polygonal domain and \(\partial \Omega\) its boundary.
- \(\tilde{q} = (q_1, \ldots, q_d)^\top\) is the vector of conserved quantities,
  \(\tilde{F} = (f_1, \ldots, f_d)^\top\) is the tensor containing the corresponding fluxes and \(\tilde{s}(\tilde{q}, \tilde{x}, t)\) is a source term.
• **Bad news:** classical solutions break down in finite time, weak solutions are not unique. For $d > 1$ there is essentially no existence, uniqueness and regularity theory.

• **Good news:** integral formulation requires less regularity, idea of *vanishing viscosity* and the Kružkov entropy solution concept can be used to select a unique weak solution.

• We restrict to the case $d = 2$ and $\bar{s}(\bar{q}, \bar{x}, t) = 0$:

\[
\begin{align*}
\bar{q}_t + \bar{f}(\bar{q})_x + \bar{g}(\bar{q})_y &= 0, \quad (x, y, t) \in \Omega \times (0, T], \\
\bar{q}(x, y, t = 0) &= \bar{q}_0(x, y), \quad (x, y) \in \Omega, \\
\bar{q}(x, y, t) &= \bar{q}_{BC}(x, y, t), \quad (x, y, t) \in \partial\Omega \times (0, T].
\end{align*}
\]
Integral Form and the Finite Volume Method

- **Idea**: integrate the differential form of the conservation law over $\mathcal{V} \times [t^n, t^{n+1}]$ and apply Gauss’ Divergence Theorem to obtain

$$\int_{t^n}^{t^{n+1}} \left( \frac{d}{dt} \int_{\mathcal{V}} \bar{q}(\vec{x}, t) \, dA + \oint_{\partial \mathcal{V}} \vec{f}(\bar{q}(\vec{x}, t)) \nu_x + \vec{g}(\bar{q}(\vec{x}, t)) \nu_y \, ds \right) \, dt = 0.$$

Using the **Fundamental Theorem of Calculus**, we get that

$$\frac{1}{|\mathcal{V}|} \int_{\mathcal{V}} \bar{q}(\vec{x}, t^{n+1}) \, dA = \frac{1}{|\mathcal{V}|} \int_{\mathcal{V}} \bar{q}(\vec{x}, t^n) \, dA$$

$$- \frac{1}{|\mathcal{V}|} \int_{t^n}^{t^{n+1}} \oint_{\partial \mathcal{V}} \vec{f}(\bar{q}(\vec{x}, t)) \nu_x + \vec{g}(\bar{q}(\vec{x}, t)) \nu_y \, ds \, dt.$$

- **Godunov’s idea**: Approximate the cell-averages of $\bar{q}$ by a piecewise-constant function, which we can evolve “exactly.”

- How do we find a “good” approximation to the flux integrals?
Godunov-type Schemes: 1D Example (I)

- Consider the 1D conservation law $\vec{q}_t + \vec{f}(\vec{q})_x = 0$.
- Introduce a mesh size $\Delta x$, and consider the Steklov sliding average
  \[ \bar{\vec{q}}(x, t) := \frac{1}{\Delta x} \int_{I_x} \vec{q}(\xi, t) \, d\xi, \quad I_x = \left\{ \xi \mid |\xi - x| \leq \frac{\Delta x}{2} \right\}. \]
- Then, integrating the conservation law over $I_x$ gives the semi-discrete form
  \[ \frac{d}{dt} \bar{\vec{q}}(x, t) + \frac{1}{\Delta x} \left[ \vec{f}(\bar{\vec{q}}(x + \frac{\Delta x}{2}, t)) - \vec{f}(\bar{\vec{q}}(x - \frac{\Delta x}{2}, t)) \right] = 0. \]
Then, integrating over the temporal interval \([t^n, t^{n+1}]\) gives the fully-discrete, conservative scheme

\[
\overline{q}(x, t^{n+1}) = \overline{q}(x, t^n) - \frac{1}{\Delta x} \left[ \int_{t^n}^{t^{n+1}} \overline{f}\left(\overline{q}(x + \frac{\Delta x}{2}, t)\right) dt - \int_{t^n}^{t^{n+1}} \overline{f}\left(\overline{q}(x - \frac{\Delta x}{2}, t)\right) dt \right].
\]

Let \(x_i = (i - 1/2)\Delta x\) be the center of each computational cell.

Then, there are two methods of sampling the F.V. formulation:

1. **(Upwind)** We sample at the cell-centers \(x_i\) at all time steps (\(\equiv\) Godunov’s idea).
2. **(Central)** We alternate between cell-interfaces \(x_{i+1/2} \equiv i\Delta x\) and the cell-centers \(x_i\).
Central Schemes in 1D (I)

- Let $w(x, t) \approx \bar{q}(x, t)$ and recall Godunov’s idea.
- **Anzats:** At each discrete time $t^n$ the approximate solution is a piecewise polynomial

$$w^n(x) := w(x, t^n) = \sum_{i=1}^{N} p^n_i(x) \chi_i(x), \quad \chi_i(x) := 1_{I_i},$$

where $p_i(x)$ is an algebraic polynomial supported on the interval $I_i \equiv I_{x_i}$, which is centered around the cell-center $x_i$.

- Sampling the sliding average equation at the cell-interfaces $x_{i+1/2}$ we obtain

$$\bar{w}^{n+1}(x_{i+1/2}) = \bar{w}^n(x_{i+1/2})$$

$$- \frac{1}{\Delta x} \left[ \int_{t^n}^{t^{n+1}} \vec{f}(w(x_{i+1}, t)) \, dt - \int_{t^n}^{t^{n+1}} \vec{f}(w(x_{i-1}, t)) \, dt \right].$$
Given $p_i^n$, we can compute exactly

$$\bar{w}^n(x_{i+1/2}) = \frac{1}{\Delta x} \left[ \int_{x_i}^{x_{i+1/2}} p_i^n(x) \, dx + \int_{x_{i+1/2}}^{x_{i+1}} p_{i+1}^n(x) \, dx \right].$$

And under the CFL condition $\max_i \rho \left( \frac{\partial \bar{f}}{\partial q}(\bar{w}^n(x_i)) \right) \Delta t < \frac{\Delta x}{2}$ the numerical fluxes can be computed with an appropriate quadrature rule, say the midpoint rule:

$$\frac{1}{\Delta x} \int_{t_n}^{t_{n+1}} \bar{f}(w(x_{i\pm1}, t)) \, dt \approx \frac{\Delta t}{\Delta x} \bar{f}(w(x_{i\pm1}, t_{n+1/2})).$$

where $t_{n+1/2} := \frac{1}{2}(t^n + t^{n+1})$. 
1. Starting with the piece-wise constant approximation to the cell average $\bar{w}^n(\cdot)$, reconstruct a piece-wise linear function $w^n(\cdot)$.

2. Compute the staggered averages $\bar{w}_{i+1/2}^{n+1}$, which gives us $\bar{w}^{n+1}(\cdot)$. 

\[ \bar{u}_{j+1/2}^{n+1} \]

\[ \bar{u}_j^n + (u_x)_j^n (x - x_j) \]

\[ \bar{u}_{j+1}^n + (u_x)_{j+1}^n (x - x_{j+1}) \]
One of the simplest (yet most robust!) second-order, nonoscillatory reconstructions is

\[ p^n_i(x) = \bar{w}^n_i + \frac{w'_i}{\Delta x} (x - x_i), \quad w'_i = \text{mm}(\bar{w}^n_{i+1} - \bar{w}^n_i, \bar{w}^n_i - \bar{w}^n_{i-1}), \]

where \( \text{mm}(a, b) \) is the \textit{minmod} (i.e., minimum-modulus) function

\[ \text{mm}(a, b) = \frac{1}{2} (\text{sgn}(a) + \text{sgn}(b)) \min(|a|, |b|). \]
1. Predict the temporal midvalues for the quadrature rule by a Taylor series expansion:

\[ \bar{w}_{i \pm 1}^{n+1/2} := w(x_{i \pm 1}, t^{n+1/2}) = \bar{w}_{i \pm 1}^n - \frac{\Delta t}{2} \frac{\partial \vec{f}}{\partial \vec{q}}(\bar{w}_{i \pm 1}^n) \frac{w'_{i \pm 1}}{\Delta x}. \]

2. Compute the new (staggered) averages

\[ \bar{w}_{i+1/2}^{n+1} = \frac{1}{2} (\bar{w}_i^n + \bar{w}_{i+1}^n) + \frac{1}{8} (w'_i - w'_{i+1}) - \frac{\Delta t}{\Delta x} \left[ \vec{f}(w_{i+1}^{n+1/2}) - \vec{f}(w_{i-1}^{n+1/2}) \right]. \]
For a scalar, one-dimensional, homogenous conservation law, the following is true:

**Theorem (B. Popov and O. Trifonov)**

Let $f$ be strictly convex and assume that the CFL condition

$$\frac{\Delta t}{\Delta x} \max |f'| \leq \kappa$$

holds with $\kappa$ sufficiently small. Then, the Nessyahu–Tadmor scheme converges to the unique (Kružkov) entropy solution of the conservation law.

Preliminaries for the 2D Godunov-type Scheme

- $\mathcal{T} = \{\tau_i\}$ is a conforming triangulation of $\bar{\Omega}$.
- $S = \{\sigma_k\}$ is the staggered grid — the “dual” of $\mathcal{T}$.
- Let $\bar{w}^n$ be a piecewise-constant function that approximates the cell averages of $\bar{q}$ on $\mathcal{T}$ at time $t^n$.

$$\bar{w}^n(x, y) = \sum_{i=1}^{N} \bar{w}^n_i \chi_i(x, y),$$

- Also, let $\bar{\omega}^n$ be the analogue of $\bar{w}^n$ on $S$.
- The piecewise-linear reconstruction of $\bar{w}^n$ will have the form

$$w^n(x, y) = \sum_{i=1}^{N} p^n_i(x, y) \chi_i(x, y),$$

$$p^n_i(x, y) = \bar{w}^n_i + \nabla p^n_i \cdot (x - x^*_i, y - y^*_i)^\top.$$
Overview of the 2D Central Scheme

1. Perform a slope-limited piecewise-linear reconstruction on $\mathcal{T}$.
   \[
   \bar{w}^n \longrightarrow w^n
   \]

2. Evolve the cell averages on the staggered grid $S$ in time.
   \[
   w^n \longrightarrow \bar{w}^n, \quad \bar{w}^n \longrightarrow \bar{w}^{n+1}
   \]

3. Project the solution from $S$ back onto $\mathcal{T}$.
   \[
   \bar{w}^{n+1} \longrightarrow w^{n+1}, \quad w^{n+1} \longrightarrow \bar{w}^{n+1}
   \]

The good, the bad and the ugly:

- No need to solve a Riemann problem at each cell interface!
- Need to define $S$ in a “reasonable” manner.
- Need to be able to perform a nonoscillatory reconstruction on $S$. 

Ivan Christov et al. (TAMU)  Nonoscillatory Central Schemes ... ULL Colloquium 17 / 25
The Minimum-Angle-Plane Reconstruction

The algorithm:

1. Given an element $\tau_i \in \mathcal{T}$ and its neighbors $\tau_{ij}$, $1 \leq j \leq m$, find all $\binom{m+1}{3}$ possible planes.

2. Find the plane that makes the smallest angle with the horizontal, and use it to find a limited gradient.

Note that

- This "genuinely 2D" limiter behaves like minmod with a UNO flavor (i.e., $\approx$ Durlofsky–Engquist–Osher but $> 1^{\text{st}}$ order near extrema).
- No particular geometry and/or connectivity is assumed in the design of the limiter, and there are no ad hoc parameters.
- Limiter works exactly the same way on $S$ as on $\mathcal{T}$. 
Evolution Through the Staggered Averages

As before ...

1 Predict the temporal midvalues for the quadrature rule with via Taylor series expansion:

\[ w(x, y, t^{n+1/2}) \approx w^n(x, y) - \frac{\Delta t}{2} \left( \frac{\partial \tilde{f}}{\partial \tilde{q}}(w^n(x, y)) \frac{\partial w^n}{\partial x} + \frac{\partial \tilde{g}}{\partial \tilde{q}}(w^n(x, y)) \frac{\partial w^n}{\partial y} \right). \]

2 Compute the new (staggered) averages:

\[ \tilde{w}^{n+1}_k \approx \tilde{w}^n_k - \frac{\Delta t}{|\sigma_k|} \int_{\partial \sigma_k} \tilde{f}(\tilde{w}(x, y, t^{n+1/2})) \nu_x + \tilde{g}(\tilde{w}(x, y, t^{n+1/2})) \nu_y \, ds. \]
The Staggered Grid

The dual elements:

1. Triangles $\Delta_i$.
2. Polygons $\Lambda_{ij}$.
3. Parallelograms $\Pi_{ij}$.

The usual CFL condition

$$\Delta t < \frac{1}{3} \cdot \min_i |\tau_i| / S_{\text{max}},$$

$S_{\text{max}}$ = fastest wave’s speed, is good enough.

In particular:

- If $|\Delta_i| = |\Pi_{ij}| = 0$, the staggered grid becomes the Voronoi diagram.
- If the maximum local speed of propagation is used, we get the analogue of Kurganov & Tadmor’s modified central differencing for triangulations.
Numerical Results for a Convex Flux

- Riemann problems for the 2D inviscid Burgers equation

\[ u_t + \left( \frac{1}{2} u^2 \right)_x + \left( \frac{1}{2} u^2 \right)_y = 0. \]

- (L) 6,272 elements and 19,041 dual elements.
- (R) 12,800 elements and 38,721 dual elements.
Riemann Problem for the Euler Equations

- \( \mathbf{q} = (\rho, \rho u, \rho v, E)^\top \), \( \tilde{\mathbf{f}}(\mathbf{q}) = (\rho u, \rho u^2 + p, \rho uv, u(E + p))^{\top} \),
- \( \tilde{\mathbf{g}}(\mathbf{q}) = (\rho v, \rho uv, \rho v^2 + p, v(E + p))^{\top} \),
- For an ideal gas: \( p = (\gamma - 1) \left[ E - \frac{1}{2} \rho (u^2 + v^2) \right] \), \( \gamma = 1.4 \).

- Very fine mesh (\( \min_{i} \text{diam}(\tau_i) = \sqrt{2}/256 \)) with CFL = 0.275.
Numerical Results for a *Nonconvex* Flux (I)

- Riemann problem for the 2D scalar equation

\[ u_t + (\sin u)_x + (\cos u)_y = 0. \]

- Adapted mesh with (only) 3,264 elements and 9,837 dual elements, CFL = \( \frac{1}{6} \).
Kurganov, Petrova & B. Popov reported that less compressive / higher order limiters (e.g., WENO5, MM2, SB) do not resolve the resulting composite wave correctly. The MAPR passes this test!

(L) 90,000 element Cartesian tensor-product mesh.
Selected Bibliography


G.-S. Jiang & E. Tadmor, “Nonoscillatory Central Schemes for Multidimensional ... ”  

A. Kurganov & G. Petrova, “Central-Upwind Schemes on Triangular Grids ... ”  

A. Kurganov, G. Petrova & B. Popov, “Adaptive Semi-Discrete Central-Upwind Schemes ... ”  