Nonoscillatory Central Schemes on Unstructured Triangular Grids for Hyperbolic Systems of Conservation Laws

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SIAM Conference on Analysis of Partial Differential Equations,
Boston, Massachusetts
July 12, 2006

*Supported by a SIAM Student Travel Award.

The Big Picture

- Motivation:
  - How should we design "genuinely multidimensional" limiters on unstructured grids?
  - Need for a simple & fast predictor algorithm for the time-dependent extension of L^1-minimization FEM (Guermond & B. Popov).
- Background:
  - High-resolution central schemes: Nessyahu & Tadmor (1992), Jiang & Tadmor (1998), Kurganov & Tadmor (2000), ...
  - Extension to unstructured grids: Arminjon *et al.* (1997).
  - Semi-discere central-upwind version: Kurganov & Petrova (2005).
- Outline of the talk:
  1. Hyperbolic systems of conservation laws and 2D central schemes.
  2. The minimum-angle-plane reconstruction.
  3. Construction of the staggered grid corresponding to a triangulation.
  4. Numerical results for equations with *convex* and *nonconvex* fluxes.

Statement of the Problem and Notation

- Consider the initial-boundary-value problem for a 2D *hyperbolic* system of conservation laws:

$$\begin{cases} \vec{u}_t + \vec{f}(\vec{u})_x + \vec{g}(\vec{u})_y = 0, & (x,y,t) \in \Omega \times (0,T], \\ \vec{u}(x,y,t=0) = \vec{u}_0(x,y), & (x,y) \in \Omega, \\ \vec{u}(x,y,t) = \vec{u}_{\mathrm{BC}}(x,y,t), & (x,y,t) \in \partial\Omega \times (0,T]. \end{cases}$$

- $\Omega \subset \mathbb{R}^2$ is the interior of a polygonal domain and $\partial\Omega$ its boundary.
- $\mathcal{T} = \{\tau_i\}$ is a *conforming* triangulation of $\bar{\Omega}$.
- $\bar{w}^n$ is a piecewise-constant approximation to the cell averages of u on $\mathcal{T}$ at time t^n.
- $\mathcal{S} = \{\sigma_k\}$ is the *staggered* grid — the "dual" of $\mathcal{T}$.
- $\bar{\mathfrak{w}}^n$ is the analogue of $\bar{w}^n$ on $\mathcal{S}$.

Overview of the 2D Central Scheme

1. Perform a slope-limited piecewise-linear reconstruction on $\mathcal{T}$.

$$\bar{w}^n \longrightarrow w^n$$

2. Evolve the cell averages on the staggered grid $\mathcal{S}$ in time.

$$w^n \longrightarrow \bar{\mathfrak{w}}^n, \quad \bar{\mathfrak{w}}^n \longrightarrow \bar{\mathfrak{w}}^{n+1}$$

3. Project the solution from $\mathcal{S}$ back onto $\mathcal{T}$.

$$\bar{\mathfrak{w}}^{n+1} \longrightarrow \mathfrak{w}^{n+1}, \quad \mathfrak{w}^{n+1} \longrightarrow \bar{w}^{n+1}$$

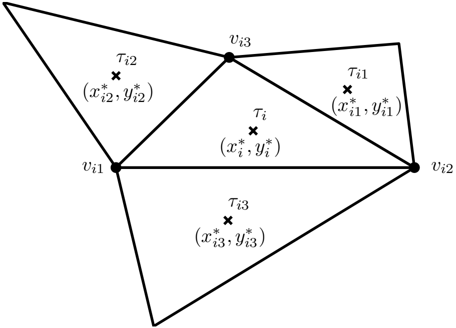
The good, the bad and the ugly:

- No need to solve a Riemann problem at each cell interface!
- Need to define $\mathcal{S}$ in a "reasonable" manner.
- Need to be able to perform a nonoscillatory reconstruction on $\mathcal{S}$.

The Minimum-Angle-Plane Reconstruction

The algorithm:

1. Given an element $\tau_i \in \mathcal{T}$ and its neighbors τ_{ij}, $1 \leq j \leq m$, find all $\binom{m+1}{3}$ possible planes.
2. Find the plane that makes the smallest angle with the horizontal, and use it to find a *limited* gradient.



Note that

- This "genuinely 2D" limiter behaves like minmod with a UNO flavor (i.e., $\approx$ Durlofsky–Engquist–Osher but $> 1^{\text{st}}$ order near extrema).
- No particular geometry and/or connectivity is assumed in the design of the limiter, and there are no *ad hoc* parameters.
- Limiter works exactly the same way on $\mathcal{S}$ as on $\mathcal{T}$.

The Staggered Grid

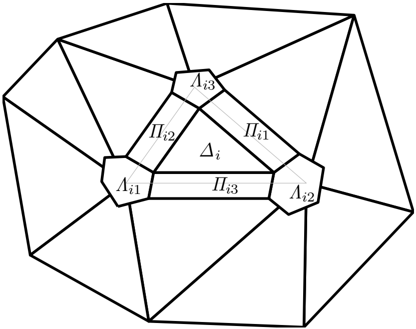
The dual elements:

1. Triangles Δ_i.
2. Polygons Λ_{ij}.
3. Parallelograms Π_{ij}.

The usual CFL condition

$$\boxed{\Delta t < \frac{1}{3} \cdot \min_i |\tau_i| / S_{\max},}$$

$S_{\max}$ = fastest wave's speed, is good enough.



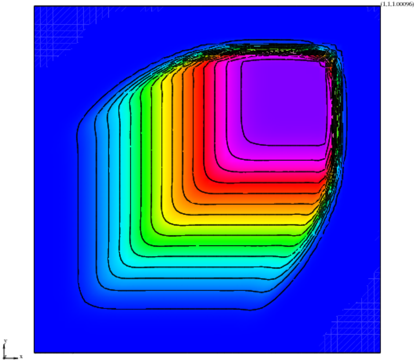
In particular:

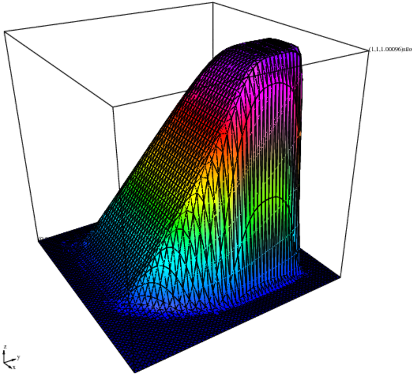
- If $|\Delta_i| = |\Pi_{ij}| = 0$, the staggered grid becomes the Voronoi diagram.
- If local speeds of propagation are used, this becomes Kurganov & Petrova's central-upwind staggered grid.

Numerical Results for a *Convex* Flux

- Riemann problem for the 2D inviscid Burgers equation

$$u_t + \left(\tfrac{1}{2}u^2\right)_x + \left(\tfrac{1}{2}u^2\right)_x = 0.$$

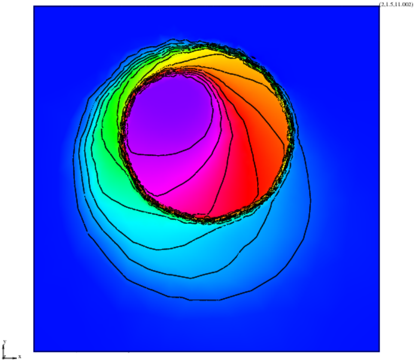


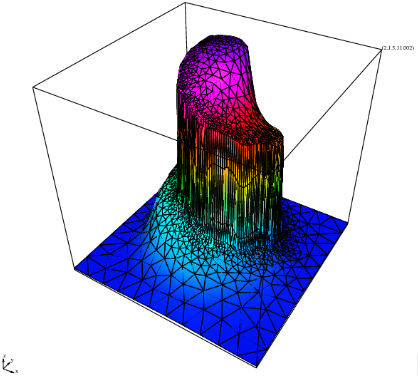


- Structured mesh with 6,272 elements and 19,041 dual elements.

Numerical Results for a *Nonconvex* Flux

- Riemann problem for the scalar equation $u_t + (\sin u)_x + (\cos u)_y = 0$.
- Kurganov, Petrova & B. Popov reported that less compressive / higher order limiters (e.g., WENO5, MM2, SB) do *not* resolve the resulting composite wave correctly. The MAPR passes this test!





- Adapted mesh with 3,264 elements and 9,837 dual elements.

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