# On the Application of Godunov-type Schemes to Conservation Laws Arising from the Equations of Nonlinear Acoustics*

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# The Big Picture

- Motivation:
  - Acoustics is *not* just the study of linearized equations!
  - There are many weakly-nonlinear models. How are they all related? Which are "better"?
  - Numerical studies of the equations of nonlinear acoustics are scarce.
- The "classical" equations of nonlinear acoustic:
  - Westervelt (1963) derived a nonlinear wave equation for the pressure of highly-directional acoustic transmitters by ignoring the viscous terms in Lighthill's (1952) turbulence stress tensor for the acoustic field.
  - Kuznetsov (1971) expanding the barotropic equation of state of an ideal gas in a power series.
- Outline of the talk:
  1. Physical setup, assumptions and the equations of motion.
  2. Weakly-nonlinear approximations.
  3. Some analytical results for a BVP.
  4. Construction of "artificial" conservation laws.
  5. Comparison of the models before and after shock-formation.

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# Governing Equations of Motion

- General assumptions and setup:
  1. The fluid is lossless ($\equiv$ inviscid) and compressible.
  2. The fluid is a perfect ($\equiv$ thermally and calorically ideal) barotropic gas.
  3. Equilibrium ($\equiv$ constant) initial conditions.
  4. Homentropic flow — i.e., the material derivative *and* gradient of the entropy vanish everywhere.
  5. 1D flow along the $x$ axis, inflow boundary at $x = 0$.
  6. Signaling boundary conditions: a "pulse" of finite duration is introduced into either the density/pressure or velocity field at $x = 0$.
- Putting all this together results in the Euler equations, which can be written as a (nondimensionalized) conservation law:

$$\begin{pmatrix} \varrho \\ \varrho u \end{pmatrix}_t + \begin{pmatrix} \epsilon \varrho u \\ \epsilon \varrho u^2 + \epsilon^{-1} \varrho^\gamma / \gamma \end{pmatrix}_x = 0,$$

where $(0 <)\epsilon \equiv u_0/c_0$ is the Mach number.

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# Weakly-Nonlinear Approximations

- Let $\phi(x,t)$ be the acoustic potential, i.e. $\nabla\phi = (u(x,t), 0, 0)$.
- The Euler equations admit the following potential formulation:

$$[1-\epsilon(\gamma-1)\phi_t]\phi_{xx} - 2\epsilon\phi_x\phi_{tx} - \phi_{tt} = \tfrac{1}{2}\epsilon^2(\gamma+1)(\phi_x)^2\phi_{xx}.$$

- One might consider the naïve weakly-nonlinear approximation:

$$[1-\epsilon(\gamma-1)\phi_t]\phi_{xx} - 2\epsilon\phi_x\phi_{tx} - \phi_{tt} = 0.$$

- Or the inviscid version of Kuznetsov's equation:

$$\phi_{xx} - [1+\epsilon(\gamma-1)\phi_t]\phi_{tt} = 2\epsilon\phi_x\phi_{xt}.$$

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# Acceleration Wave Analysis

- An *acceleration wave* is a jump discontinuity in at least one of the derivatives of the primitive variable (e.g., $\varrho$).
- The basic idea: apply the method of characteristics at the *wavefront* and derive and expression for the amplitude of the jump as a function of time.
- *Blow-up* of the acceleration wave ($\equiv$ the jump amplitude $\to \infty$) means a shock formed in the respective primitive variable.
- Pros: gives some theoretical results, which can be confirmed numerically, and can be used in evaluating the models.
- Cons: acceleration wave analysis ignores all other effects that lead to shock-formation (e.g., nonlinear steepening).
- For all of the equations and the BVP considered here the blow-up time is the same: $t_\infty = 2c_0/\{(\gamma+1)[u_t]_{t=0}\}$.

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# Potential Equation → System of Conservation Laws

- From our assumptions, we know that $\boxed{\phi_x = u}$ and $\boxed{\phi_t = (1-\varrho)/\epsilon}$.
- So the (potential) LWE can be rewritten as

$$\left[(\gamma+2)\varrho - \tfrac{1}{2}(\gamma+1)\varrho^2\right]_t + \epsilon u_x = 0,$$
$$u_t + \epsilon^{-1}\varrho_x = 0.$$

- But this is not yet a conservation law! So, let

$$\tilde{\varrho} \stackrel{\text{def}}{=} (\gamma+2)\varrho - \tfrac{1}{2}(\gamma+1)\varrho^2, \text{ s.t.}$$
$$\varrho = \frac{1}{\gamma+1}\left[\gamma + 2 \mp \sqrt{(\gamma+2)^2 - 2(\gamma+1)\tilde{\varrho}}\right].$$

- Finally, we get

$$\begin{pmatrix} \tilde{\varrho} \\ u \end{pmatrix}_t + \begin{pmatrix} \epsilon u \\ -[\epsilon(\gamma+1)]^{-1}\sqrt{(\gamma+2)^2 - 2(\gamma+1)\tilde{\varrho}} \end{pmatrix}_x = 0.$$

- The other two weakly-nonlinear equations can be converted into conservation laws in the same manner.

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# The MUSCL–Hancock Scheme

- Consider the generic 1D hyperbolic system of conservation laws:

$$\mathbf{Q}_t + \mathbf{F}(\mathbf{Q})_x = 0.$$

- Introduce the conservative discretization:

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n + \frac{\Delta t}{\Delta x}\left(\mathbf{F}^n_{i-\frac{1}{2}} - \mathbf{F}^n_{i+\frac{1}{2}}\right).$$

- Perform a (linear) MUSCL extrapolation of the cell-interface values:

$$\mathbf{Q}_i^{\mathrm{L,R}} = \mathbf{Q}_i \mp \tfrac{1}{2}\,\mathrm{minmod}\left(\mathbf{Q}_i - \mathbf{Q}_{i-1}, \mathbf{Q}_{i+1} - \mathbf{Q}_i\right).$$

- Use predictor for the cell-interface temporal midvalues:

$$\overline{\mathbf{Q}}_i^{\mathrm{L,R}} = \mathbf{Q}_i^{\mathrm{L,R}} + \frac{1}{2}\frac{\Delta t}{\Delta x}\left[\mathbf{F}(\mathbf{Q}_i^{\mathrm{L}}) - \mathbf{F}(\mathbf{Q}_i^{\mathrm{R}})\right].$$

- Finally, use $\overline{\mathbf{Q}}_i^{\mathrm{R}}$ and $\overline{\mathbf{Q}}_{i+1}^{\mathrm{L}}$ to set up a generalized Riemann problem at each cell interface $x_{i+\frac{1}{2}}$, and the Harten–Lax–van Leer approximate Riemann solver to obtain the flux $\mathbf{F}^n_{i+\frac{1}{2}}$ $(-1 \le i \le N-1)$.

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# Numerical Results for $t < t_\infty$

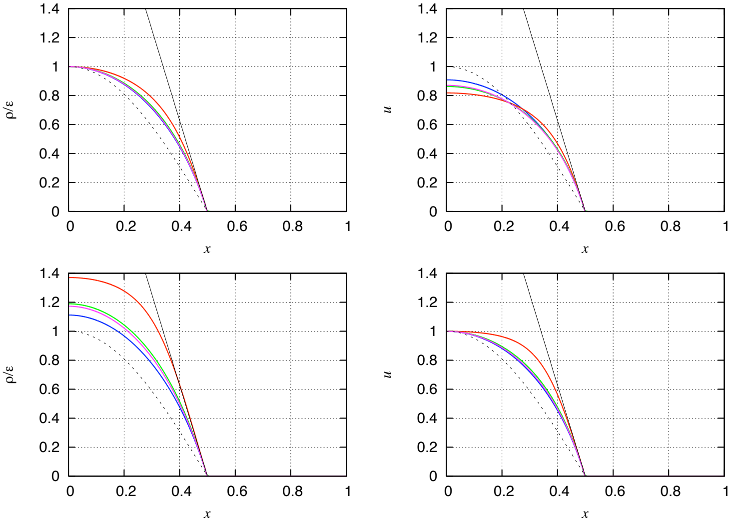


Figure: Top row: density BC; bottom row: velocity BC; $t = 0.5$.

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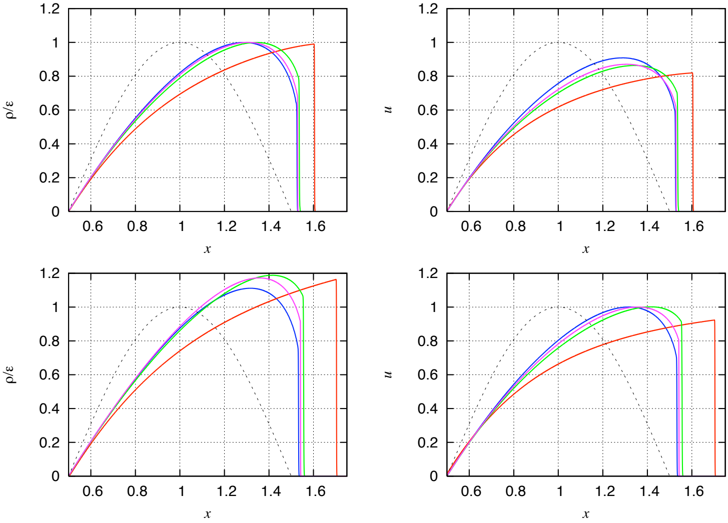


Figure: Top row: density BC; bottom row: velocity BC; $t = 1.5$.

# Conclusions and Outlook

- So, what's a "good" weakly-nonlinear approximation? Well, it's hard to say ... no theoretical results but the numerics are illuminating.
- It is clear, however, that if one *must* use a weakly-nonlinear approximation, then the "naïve" one is superior to the "classical" ones.
- Black-box solver approach was crucial for the numerical simulations. The MUSCL–Hancock scheme with the HLL solver worked well, but a *central* scheme would be better (especially in 2D).
- Further theoretical work is needed for the BVP — use the method of characteristics beyond the wavefront and compute the time of shock-formation.

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# Conclusions and Outlook

- So, what's a "good" weakly-nonlinear approximation? Well, it's hard to say ... no theoretical results but the numerics are illuminating.
- It is clear, however, that if one *must* use a weakly-nonlinear approximation, then the "naïve" one is superior to the "classical" ones.
- Black-box solver approach was crucial for the numerical simulations. The MUSCL–Hancock scheme with the HLL solver worked well, but a *central* scheme would be better (especially in 2D).
- Further theoretical work is needed for the BVP — use the method of characteristics beyond the wavefront and compute the time of shock-formation.