Two case studies in nonlinear Fourier analysis: Ocean internal solitary waves and the Zabusky–Kruskal solitons

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1. Introduction.
2. The periodic, inverse scattering transform in action: Osborne's nonlinear Fourier analysis.
3. Internal solitary waves in the ocean: Analysis using the periodic, inverse scattering transform [in collaboration with Drs. S.A. Chin-Bing (Acoustics Division), A.C. Warn-Varnas (Oceanography Division), NRL-SSC and J.A. Hawkins (Planning Systems Inc.)].
4. Hidden solitons in the Zabusky–Kruskal experiment: Analysis using the periodic, inverse scattering transform.
5. Closure.

The Big Picture

- Motivation:
  - Linear Fourier analysis is ineffective for nonlinear physical phenomena.
  - Nonlinearity is crucial in real-world problems, e.g., soliton formation.
- Background:
  - ZK 1965: discovery of solitons emerging from harmonic excitation.
  - GGKM 1967 ... AKNS 1973 ... Flaschka & McLaughlin / Dubrovin, Matveev & Novikov 1976: inverse scattering transform (IST).
  - Osborne ca.1980–today: Applied periodic IST for KdV to the analysis of oceanographic data: nonlinear Fourier analysis.
  - Salupere et al. 1994–2005: Found soliton ensembles, patterns of trajectories and hidden solitons in simulations of periodic KdV.

Featured Example

- Korteweg–de Vries equation is a paradigm in solitonics:

$$\text{PDE: } \eta_t + c_0\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = 0, \quad (x,t) \in [0,L] \times (0,\infty);$$

$$\text{IBC: } \begin{cases} \eta(x,t=0) = \eta_0(x), & x \in [0,L], \\ \eta(x+L,t) = \eta(x,t), & (x,t) \in [0,L] \times [0,\infty). \end{cases}$$

- For long surface waves over shallow water, for example, we have

$$\underbrace{c_0 = \sqrt{gh},}_{\text{speed of linear waves}} \qquad \underbrace{\alpha = \frac{3c_0}{2h},}_{\text{nonlinearity coeff.}} \qquad \underbrace{\beta = \frac{c_0 h^2}{6},}_{\text{dispersion coeff.}} \qquad \underbrace{\lambda = \frac{\alpha}{6\beta}.}_{\text{nl-to-disp. ratio for PIST}}$$

Fourier Analysis of Linear PDEs

Consider the linearized KdV equation (i.e., $\alpha = 0$):

$$\eta_t + c_0\eta_x + \beta\eta_{xxx} = 0.$$

We can find its exact solution via the Fourier Transform:

1. "Discrete Fourier Transform" (DFT): Find the spectrum $\{c_j; \phi_j\}$: $c_j = \sqrt{a_j^2 + b_j^2}$, $\phi_j = \arctan(-b_j/a_j)$, where a_j and b_j are the usual Fourier coefficients of $\eta(x, 0)$.
2. "Inverse Discrete Fourier Transform" (IDFT): Construct the solution from the spectrum $\{c_j; \phi_j\}$ as a Fourier series:

$$\eta(x,t) = \frac{c_0}{2} + \sum_{j=1}^{N} c_j \cos(k_j x - \omega_j t + \phi_j),$$

where $\omega_j = c_0 k_j - \beta k_j^3$ from the dispersion relation for the PDE.

"Fourier Analysis" of Integrable Nonlinear PDEs

Going back to the KdV equation in the moving frame:

$$\eta_t + c_0\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = 0,$$

we can find its exact solution via the Scattering Transform:

1. Direct Scattering Transform (DST): Solve associated Schrödinger EVP

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \lambda\eta(x,0)\right]\psi = E\psi$$

   to get the discrete spectrum, aka "scattering data," $\{\mathcal{E}_j\} \cup \{\mu_j^0\}$.

2. Inverse Scattering Transform (IST): Construct solution from the scattering data as a nonlinear Fourier series using hyperelliptic functions or Riemann Θ-function.

⋆ Steps ① & ② constitute the periodic, inverse scattering transform (PIST) for the KdV eq.

The Nonlinear Fourier Series

- In terms of the hyperelliptic (aka Abelian) functions we have

$$\eta(x,t) = \frac{1}{\lambda}\left\{2\sum_{j=1}^{N}\mu_j(x,t) - \sum_{j=1}^{2N+1}\mathcal{E}_j\right\}.$$

  $\star$ All nonlinear waves and their interactions in a linear superposition!

- Small amplitude limit, $\max_{x,t}|\mu_j(x,t)| \ll 1$,
  $\mu_j(x,t) \sim \cos(k_j x - \omega_j t + \phi_j) \Rightarrow$ recover Fourier series!
- No interactions (only one wave, $N = 1$),
  $\mu_1(x,t) \sim \mathrm{cn}^2(k_1 x - \omega_1 t + \phi_1 | m_1)$ (cnoidal wave).
- Amplitudes of nonlinear oscillations are given by

$$A_j = \begin{cases} \frac{2}{\lambda}(\mathcal{E}_{\mathrm{ref}} - \mathcal{E}_{2j}), & \text{for solitons;} \\ \frac{1}{2\lambda}(\mathcal{E}_{2j+1} - \mathcal{E}_{2j}), & \text{otherwise (radiation).} \end{cases}$$

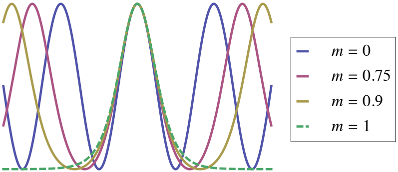
Of Moduli and Wave Numbers

- In the hyperelliptic representation, we have $k_j = 2\pi j/L$ as in Fourier theory (the wave numbers are commensurable).
- We can compute the elliptic modulus m_j ("soliton index") of the hyperelliptic functions from the discrete spectrum as

$$m_j = \frac{\mathcal{E}_{2j+1} - \mathcal{E}_{2j}}{\mathcal{E}_{2j+1} - \mathcal{E}_{2j-1}}, \qquad 1 \leq j \leq N.$$

Then, the oscillations fall into two generic categories:

  - $m_j \gtrsim 0.99 \Rightarrow$ solitons:
    $\mathrm{cn}^2(x|m_j = 1) \equiv \mathrm{sech}^2(x)$,
  - $m_j \ll 1.0 \Rightarrow$ radiation:
    $\mathrm{cn}^2(x|m_j = 0) \equiv \cos^2(x)$.



Numerics of the DST (I)

- Introduce an "appropriate" basis of eigenfuctions for this e-value problem with periodic potential and appeal to Floquet's theorem to obtain

$$\mathbf{\Phi}(x+L,E) = \boldsymbol{\alpha}(x,E)\mathbf{\Phi}(x,E),$$

where $\boldsymbol{\alpha}$ is the monodromy matrix.

- The main spectrum is $E = \mathcal{E}_j$ s.t. $\Delta(\mathcal{E}_j) := \frac{1}{2}\operatorname{tr}\boldsymbol{\alpha}(x,\mathcal{E}_j) = \pm 1$.
  - Δ is the Floquet discriminant.
- The auxiliary spectrum is $E = \mu_j$ s.t. $\alpha_{21}(x,\mu_j) = 0$.
- **Problem:** Need a numerically-accessible version of $\boldsymbol{\alpha}$ from which to compute the discrete spectrum $\{\mathcal{E}_j\} \cup \{\mu_j\}$ of our Schrödinger eigenvalue problem.

Numerics of the DST (II)

- Rewrite the Schrödinger EVP a 1st-order system for $\Psi := (\psi, \psi_x)^\top$:

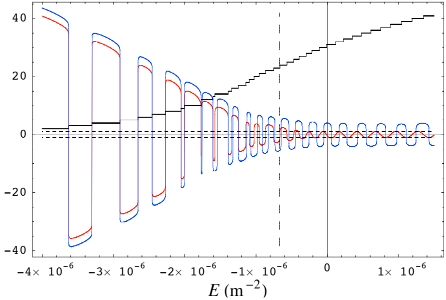
$$\frac{\mathrm{d}}{\mathrm{d}x}\Psi(x,E) = \mathbf{B}(x,E)\Psi(x,E), \quad \mathbf{B}(x,E) = \begin{pmatrix} 0 & 1 \\ -\lambda\eta(x,0) - E & 0 \end{pmatrix},$$

- $\eta(x,0)$ is given as a discrete data set at $x_i := i\Delta x$, $0 \le i \le M-1$.
- Formal integration $\Rightarrow$ $\Psi(x_{i+1}, E) = \exp\{\int_{x_i}^{x_{i+1}} \mathbf{B}(\xi, E)\,\mathrm{d}\xi\} = \mathrm{e}^{\Delta x \mathbf{B}(x_i,E)}\Psi(x_i,E)$.
- Iterating the latter:

$$\Psi(x+L,E) = \underbrace{\mathbf{M}(x,E)}_{=\prod_{i=M-1}^{0} \mathrm{e}^{\Delta x \mathbf{B}(x_i,E)}} \Psi(x,E) \quad \forall x \in [0,L].$$

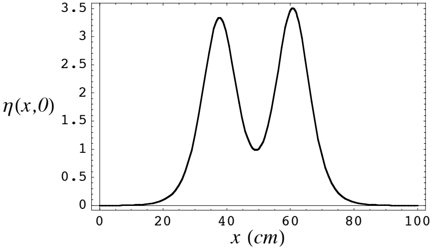
- $\therefore$ $\mathbf{M}$ is a numerical approximation to the monodromy matrix $\boldsymbol{\alpha}$.
  - Now find $\frac{1}{2}\operatorname{tr}\mathbf{M}(E) = \pm 1$ and $M_{21}(E) = 0$; see (I.C.C., *Math. Comput. Simulat.*, 2009) for numerical issues.

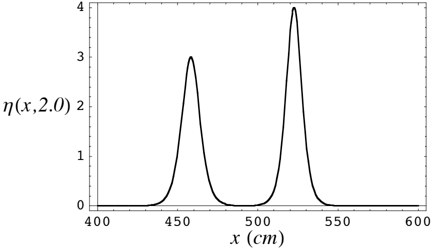
Visualizing the Numerics of the DST ("Floquet Diagrams")



- Zero crossings of $M_{21}(E)$ (in blue) determine number of degrees of freedom in DST spectrum; each crossing is a μ_j value.
- Consecutive $+1/+1$ and $-1/-1$ crossings of $\Delta(E)$ (in red) determine open bands of DST spectrum;
  each crossing is an $\mathcal{E}_j$ value.

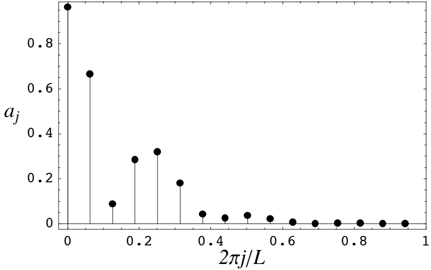
Elementary Example: KdV 2-Soliton

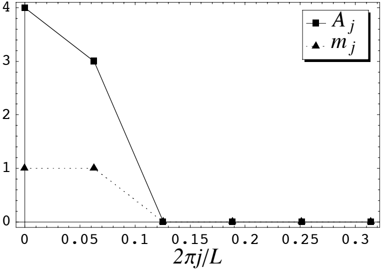




- **Signal to analyze**: infinite-line Hirota 2-soliton (at $t = 0.15$ s, for convenience); $\eta(x + 100\,\text{cm}, 0) \approx \eta(x, 0)$.
- (Left) An infinite-line Hirota 2-soliton (at $t = 0.15$ s, for convenience); $\eta(x + 100\,\text{cm}, 0) \approx \eta(x, 0)$.
- (Right) Same 2-soliton solution but at $t = 2.0$ s, note the amplitudes of the two solitons are 3.0 cm and 4.0 cm.

Comparison of the DFT & DST Analyses



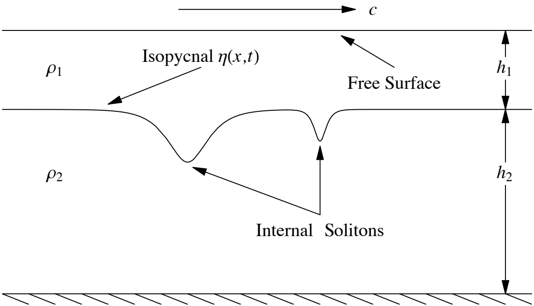


- The DFT finds ≈ 16 normal modes.
- (Left) The DFT finds ≈ 16 normal modes.
- (Right) The DST finds *exactly* 2.
- **N.B.**: When data is inherently nonlinear (e.g., solution of KdV), DST offers a better representation than DFT.

A model for internal solitary waves (Osborne & Burch, 1980)

KdV Eq.: $\eta_t + c_0\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = 0, \quad (x,t) \in [0,L] \times (0,\infty);$

$$\text{IBC: } \begin{cases} \eta(x, t=0) = \eta_0(x), & x \in [0,L], \\ \eta(x+L, t) = \eta(x,t), & (x,t) \in [0,L] \times [0,\infty). \end{cases}$$



The coefficients are now

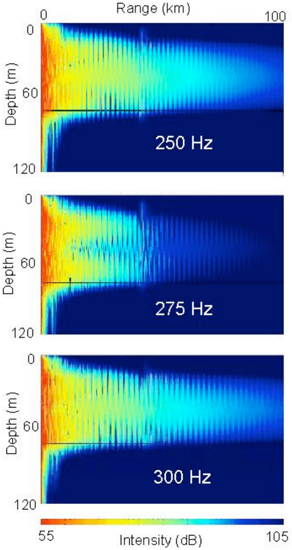
$$c_0 \simeq \sqrt{g\frac{\Delta\rho}{\rho}\left(\frac{h_1h_2}{h_1+h_2}\right)},$$

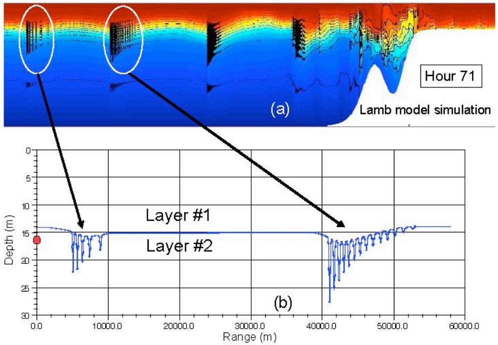
$$\alpha \simeq \frac{-3c_0}{2}\left(\frac{h_2-h_1}{h_1h_2}\right),$$

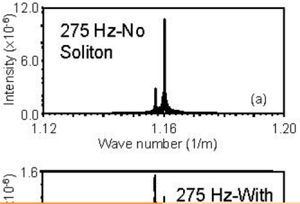
$$\beta \simeq \frac{1}{6}c_0h_1h_2,$$

$$\mathrm{Ur} \propto \frac{\eta_{\max}/h_2}{(h_2/L)^2}$$

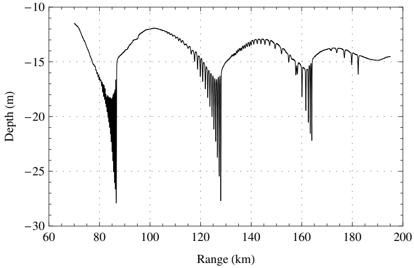
Internal Solitary Waves ⇒ "Anomalous" Acoustic Loss?

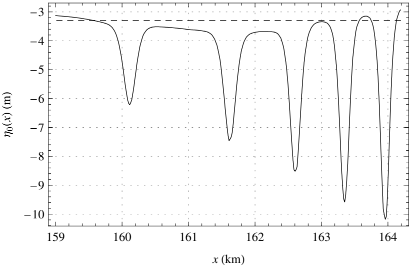






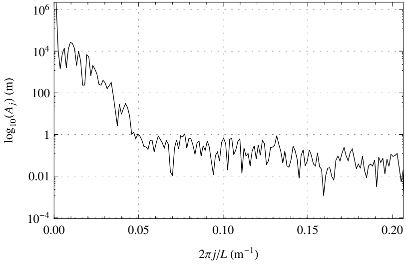
Internal Solitary Waves in the Yellow Sea



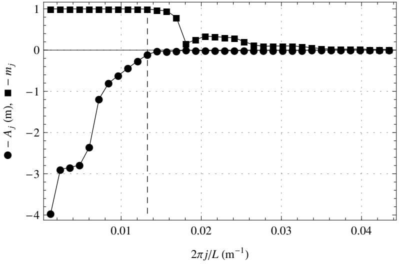


- (Left) Numerical simulations of internal-solitary-wave generation by bottom topography in the Yellow Sea via nonhydrostatic Lamb model (Chin-Bing et al., 2003); $\sigma_t = 22.0$ isopycnal located at a depth of ≈ 12 m. [Depth measured from the sea bottom.]
- (Right) Third wave packet from left to be analyzed; Ur ≈ 27. [Depth measured from the undisplaced isopycnal.]

Fourier Analysis of the Internal Solitary Waves



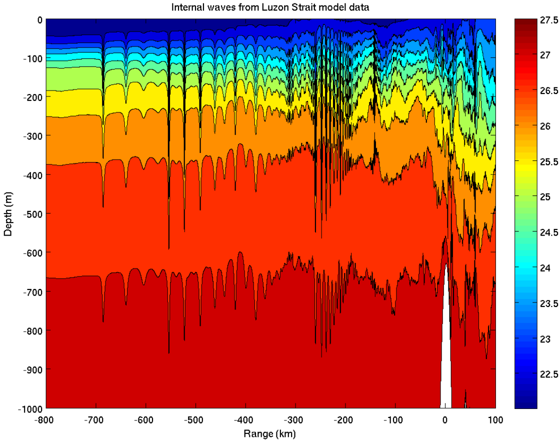
(a) DFT spectrum.



(b) The DST spectrum.

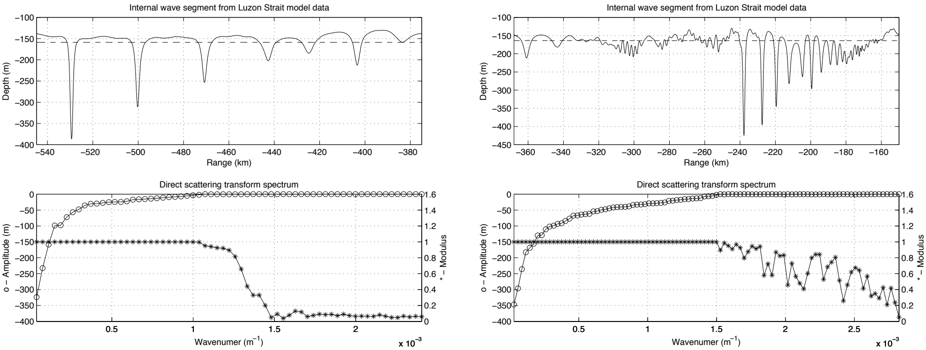
- Three things to note about the DFT vs. DST spectrum:
  - the narrower (and distinct) range of wave numbers predicted,
  - the nonlinear non-soliton waves present in the spectrum,
- The DST can find solitons in the data set. [No guess work.]
  - DST k-range agrees with empirical one based on DFT.
  - Resonances were found as predicted by Z,Z&R formula.

Internal Solitary Waves in the Strait of Luzon



Simulation of internal-solitary-wave generation by bottom topography in the Luzon Strait (Hawkins, Warn-Varnas & I.C.C., 2008).

A Windowed Nonlinear Fourier Transform?



(a) Ur = 3.209, 28 solitons (b) Ur = 5.718, 52 solitons

- Solitary wave packets evolve as they propagate away from the sill.
- Performing a NFA of each and studying how the DST spectrum has changed could provide insight into soliton "aging."

Statement of the Problem and Notation

- Again, we study the periodic IBVP for the KdV equation:

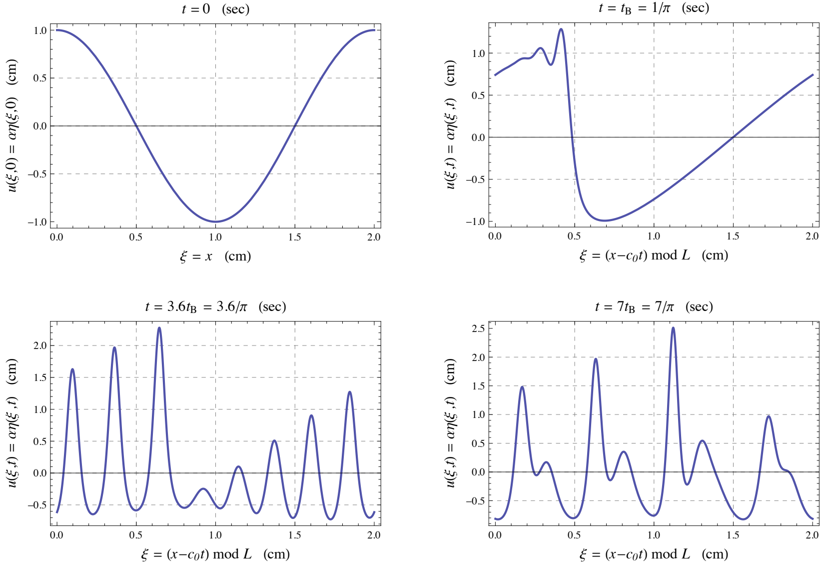
$$\text{PDE: } \eta_t + c_0\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = 0, \quad (x,t) \in [0,L] \times (0,\infty);$$

$$\text{IBC: } \begin{cases} \eta(x, t=0) = \eta_0(x), & x \in [0,L], \\ \eta(x+L,t) = \eta(x,t), & (x,t) \in [0,L] \times [0,\infty). \end{cases}$$

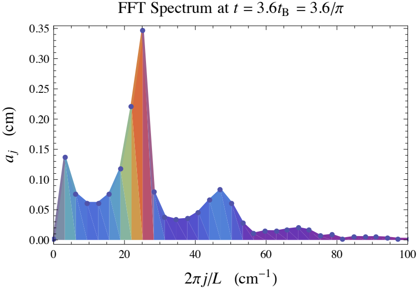
- Now, we use the IC of Zabusky & Kruskal (1965):

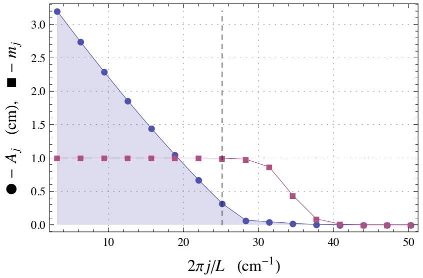
$$\eta_0(x) = A\sin(\omega x), \quad x \in [0,L], \quad \omega = \frac{2\pi}{L}n, \quad n \in \mathbb{Z}.$$

Soliton Formation from a Harmonic Excitation [ZK 1965]



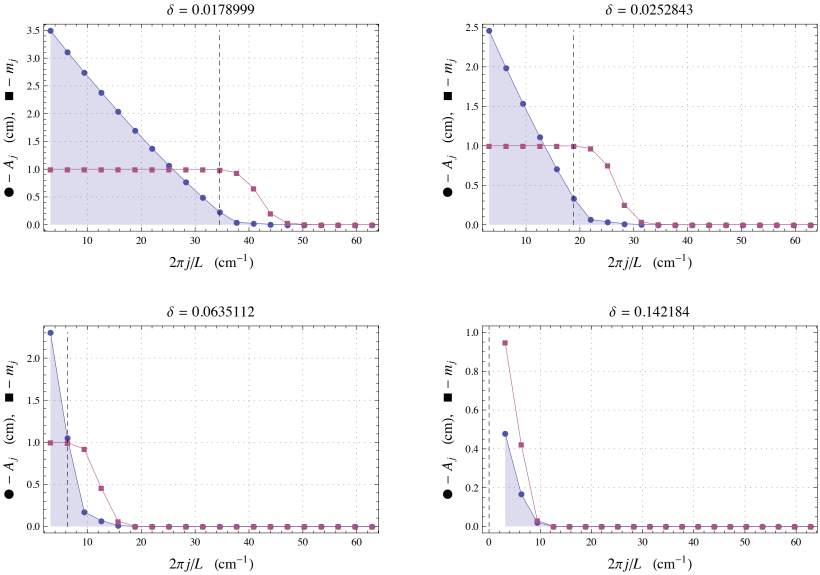
Back to Soliton Formation; $\delta = 0.022$ [ZK 1965]





- (Left) The DFT spectrum gives us 30+ significant normal modes.
- (Right) The DST gives 8 (well-defined) soliton modes (=ZK 1965), other nonlinear waves, and radiation.
- **N.B.**: $\exists$ 4 non-soliton nonlinear waves with nontrivial amplitudes.
- Amplitude curves approach of Salupere *et al.* (1994,1996) predicts 12 waves for this case.

How Many Solitons in a Cosine Wave?



A Reference Level for the Periodic Problem

- Shift in the reference (zero) level of solitons $\equiv$ shift in the energy level of the last soliton's band gap edge with respect to $E = 0$ (Osborne & Bergamasco, 1986), so:

$$\mathcal{E}_{\text{ref}} = \mathcal{E}_{2j^*+1} - 0, \qquad j^* \text{ is biggest } j \text{ s.t. } m_j \geq 0.99.$$

- From $\mathcal{E}_{\text{ref}}$ we can determine the solitons' reference level and reference wavenumber as

$$u_{\text{ref}} = \alpha\eta_{\text{ref}}, \qquad \eta_{\text{ref}} = -\mathcal{E}_{\text{ref}}/\lambda, \qquad k_{\text{ref}} = 2\pi j^*/L.$$

δ	1.0	0.318	0.142	0.0635	0.0253	0.022	0.0179
u_{ref}	1.29e-4	7.80e-4	0.337	-0.483	-0.555	-0.720	-0.844

- Salupere et al. (1996) choose u_{ref} *a priori*, this affects the predicted # of solitons.
- PIST gives a rigorous way to calculate u_{ref}. [No guess work.]

Summary & Conclusions

- Nonlinear Fourier analysis based on the periodic, inverse scattering transform for the KdV equation is a powerful tool for data analysis.
- Offers significant insight into coupled ocean-acoustic dynamics and the "aging" of wave packets created by tidal forcing over topography.
  - Justifies "internal solitary wave" = "internal soliton" dichotomy.
  - Supports current empirical methods and mode-conversion formula.
- Can also address hidden solitons in the Zabysky–Kruskal experiment using PIST and corroborate Salupere et al.'s numerical results.
  - All visible waves are indeed solitons;
  - Other hidden (nonlinear) modes are not exactly solitons.
  - Soliton reference level for periodic case has nontrivial dependence on dispersion parameter.

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