Chaotic granular mixing in quasi-two-dimensional tumblers: Streamline jumping, piecewise isometries and strange eigenmodes

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Dynamical Systems Framework for Tumbled Granular Flow

- Kinematic approach to mixing in a quasi-2D tumbled granular flow leads to the dynamical system
  \[ \frac{d}{dt}(\mathbf{x}) = \begin{cases} \nu_x(x, y, t), & y > -\mu(x, t); \\ \omega_x(x, y, t) + q \mu(x, t) - \mu(x, t), & \text{otherwise}. \end{cases} \]

- From depth-averaged streamline velocity \( \bar{v}_z \neq \bar{v}_z(x) \) & conservation of mass (Khurshid et al., Chaos, 1999) we get
  \[ \nu_x(x, y, t) = 2\nu_x(x, t) + y/(\nu_x(x, t)), \quad \nu_x(x, y, t) = -\omega_x(x, y, t), \quad \omega_x(x, y, t) = \omega_x(x, t)^2 / (|\omega_x(x, t)|). \]

- Streamline crossing criterion: to achieve chaotic mixing, superimposed streamlines at two different times should intersect.
- Changing length and density of flowing layer \( \Rightarrow \) streamline crossing in a quasi-2D non-circular tumbler.
- Geometric similarity: \( \epsilon = \omega_x(x, t)/\omega_x(x, t) \Rightarrow \) const. \( \Rightarrow \) flowing layer adjusts instantaneously to changes in container’s orientation.
- \( \bar{v}_z \approx 1 \) to 12 particle diameters \( \Rightarrow \epsilon \approx 1. \)
- If streamline crossing \( \Rightarrow \) were the only mixing mechanism, then the flow should become trivial as \( \epsilon \rightarrow 0. \) However, this is not true. Why?

Streamline Jumping: A New Mixing Mechanism

\[ \lim_{\epsilon \rightarrow 0} \bar{v}_z = 0 \Rightarrow \text{flowing layer collapses onto the free surface becoming infinite}. \]

- The average speed of a particle on the free surface becomes infinite:
  \[ \bar{v}_z = \frac{1}{3} \sqrt{\frac{1}{4} - \frac{1}{4} \pi \epsilon^2 - \frac{1}{4} \epsilon^4}. \]

- For consistency \( \bar{v}_z = -\bar{v}_z \) when crossing the flowing layer, \( \bar{v}_z = -\bar{v}_z + 2\bar{v}_z \) w.r.t. \( \bar{v}_z \).

Limiting Dynamics as a Piecewise Isometry

- The limiting \( (\epsilon = 0) \) dynamical system is a piecewise isometry (PWI) \( \Rightarrow \) a type of discontinuous dynamical system studied only recently (Goetz, DCDS, 1996).
- PWIs exhibit the usual nonlinearity dynamics (periodic points, fractal structure, global attractors, etc.) but there is no stretching and folding (‘smea horsemen’) or chaos in the usual sense.
- The PWI is an affine transformation \( \Omega(t_1, t_2) : \mathbb{T}^2 \mathbb{R} \times \mathbb{T}^2 \mathbb{R} \mathbb{R} \) that can be written as
  \[ \Omega(t_1, t_2) = P \mathbb{R} Q(t_1, t_2) = \begin{pmatrix} \cos(\phi_2 - \phi_1) & -\sin(\phi_2 - \phi_1) \\ \sin(\phi_2 - \phi_1) & \cos(\phi_2 - \phi_1) \end{pmatrix} (\begin{pmatrix} t_1 + t_2 \\ t_1 - t_2 \end{pmatrix}). \]
- Cutting and shuffling dynamics: \( \Omega \) “shuffles” by mapping each point in the flowing layer to a new location, \( R \times T \) “cuts” by reflecting & translating points along the flowing layer. This leads to spreading of material points in the continuum in the absence of stretching by shear flow.

Finite-Time Lyapunov Exponents and Manifold Structure

- Mixing and transport are affected by the underlying invariant structures of the flow, e.g., manifolds.
- So, consider the (largest) finite-time Lyapunov exponent (FTLE) field of the flow
  \[ \nu(x, y, t) = \frac{1}{|\nu_x(x, y, t)|} \ln \frac{1}{|\nu_x(x, y, t)|}. \]

- A Lagrangian coherent structure (LCS) is a ridge of \( \sigma \), for a qualitatively analysis, can identify ridges with the darkest areas in the figure.
- Mass flux across an LCS is negligible (Shadden et al., Physica D, 2005) \( \Rightarrow \) LCSs are finite-time analogues to the stable/unstable manifolds of the flows with arbitrary time-dependence.

Important point: manifold structure of the \( \epsilon = 0 \) system is the template or skeleton for all \( \epsilon < 1 \Rightarrow \) streamline jumping is the predominant mixing mechanism here (Christov et al., Chaos, 2010).

Evendmodes of a Tumbled Granular Flow

- The concentration \( C \) of a diffusive tracer being mixed is governed by an advection diffusion eq.
  \[ \partial_t C(x, t) = -\nabla \cdot (C(x, t) - \bar{u}(x, t)). \]
- Asymptotic tracer distribution is determined by the dominant (‘strong’) eigenmodes (Pierrhembert, C.S., D.E., 1994). In time-periodic flows, introduce Floquet operator \( F \) s.t. \( C(x, t + T) = F^n C(x, t) \).
- We can find the eigenmodes of the Floquet operator \( F \) by the simplified mapping method (Singh et al., Comput. Ch.E., 2009). Those with eigenvalues \( |\lambda| \approx 1 \) persist for a very long time.
- For a monodisperse granular mixture (no segregation), strange eigenmodes form the asymptotic mixing pattern, but they give the segregation (unmixing) pattern for a bidisperse mixture.

Summary and Open Questions

- The \( \epsilon = 0 \) dynamics form the “skeleton,” and mixing occurs via streamline jumping.
- The limit reveals new application of PWIs; LCSs show the manifold structure of this non-smooth flow.
- Strange eigenmodes exist in granular flows and determine the asymptotic advection pattern.
- In 3D, what happens when multiple-axes rotation protocols are combined with streamline jumping?
- Can we predict granular mixing a priori based on container geometry and PWIs? Can mixing protocols be designed to suppress the strange eigenmodes?
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