Internal solitary waves in the ocean: Analysis using the periodic, inverse scattering transform

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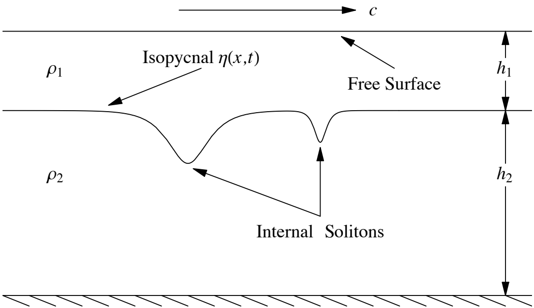
The Big Picture

- Motivation:
  - Linear Fourier analysis often proves ineffective for the analysis of inherently nonlinear physical phenomena.
  - Nonlinearity plays a crucial part in real-world problems.
- Background:
  - GGKM 1967 ... AKNS 1973 ... Flaschka, McLaughlin / Dubrovin, Matveev, Novikov 1976: Developed a general method for solving nonlinear evolution equations termed the (inverse) scattering transform.
  - Osborne ca.1980–2000: Application of the scattering transform for the periodic KdV eq. to the analysis of oceanographic data — the so-called nonlinear Fourier analysis.
- Outline of the talk:
  1. Modeling propagation of internal solitons in the ocean: the Korteweg–de Vries equation.
  2. The inverse scattering transform as a nonlinear Fourier analysis.
  3. Numerics of the discrete scattering transform.
  4. Analysis of internal solitary wave trains in the Yellow Sea using the discrete scattering transform.

A model for internal solitary waves (Osborne & Burch 1980):

KdV Eq.: $\eta_t + c_0\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = 0, \quad (x,t) \in [0,L] \times (0,\infty);$

$$\text{IBC: } \begin{cases} \eta(x, t=0) = \eta_0(x), & x \in [0,L], \\ \eta(x+L,t) = \eta(x,t), & (x,t) \in [0,L] \times [0,\infty). \end{cases}$$



The physical parameters are

$$c_0 \simeq \sqrt{g\frac{\Delta\rho}{\rho}\left(\frac{h_1 h_2}{h_1+h_2}\right)},$$

$$\alpha \simeq \frac{-3c_0}{2}\left(\frac{h_2-h_1}{h_1 h_2}\right),$$

$$\beta \simeq \frac{1}{6}c_0 h_1 h_2,$$

$$\Delta\rho := \rho_2 - \rho_1,\ \rho \simeq \rho_1 \simeq \rho_2.$$

Fourier Analysis of Linear PDEs

Consider the linearized KdV equation (i.e., $\alpha = 0$):

$$\eta_t + c_0\eta_x + \beta\eta_{xxx} = 0.$$

We can find its exact solution in terms of a Fourier Series:

1. "Discrete Fourier Transform" (DFT): Find the spectrum $\{c_j; \phi_j\}$: $c_j = \sqrt{a_j^2 + b_j^2}$, $\phi_j = \arctan(-b_j/a_j)$, where a_j and b_j are the usual Fourier coefficients.
2. "Inverse Discrete Fourier Transform" (IDFT): Construct the solution to the linear PDE from the spectrum $\{c_j; \phi_j\}$:

$$\eta(x,t) = \frac{a_0}{2} + \sum_{j=1}^{N} c_j \cos(k_j x - \omega_j t + \phi_j),$$

where $\omega_j = c_0 k_j - \beta k_j^3$ from the dispersion relation for the PDE.

"Fourier Analysis" of Integrable Nonlinear PDEs

Going back to the (nonlinear) KdV equation:

$$\eta_t + c_0\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = 0,$$

we can find its exact solution via the Scattering Transform:

1. Direct Scattering Transform (DST): Solve the associated Schrödinger eigenvalue problem:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \lambda\eta(x,0)\right]\psi = E\psi,$$

   where $\lambda = \frac{\alpha}{6\beta}$ is the nonlinearity to dispersion ratio, to get the discrete spectrum, aka "scattering data," $\{E_j; \mu_j\}$.

2. Inverse Scattering Transform (IST): Construct the nonlinear Fourier series from the spectrum $\{E_j; \mu_j\}$ using hyperelliptic functions or the Riemann Θ-function.

⋆ Together these two steps constitute the periodic, inverse scattering transform (PIST) for the KdV eq.

The Nonlinear Fourier Series

- In terms of the hyperelliptic (aka Abelian) functions we have

$$\eta(x,t) = \frac{1}{\lambda}\left(-E_1 + \sum_{j=1}^{N} 2\mu_j(x,t) - E_{2j} - E_{2j+1}\right).$$

  All nonlinear waves and their interactions are obtained from this linear superposition!
- In the small amplitude limit, $\max_{x,t} |\mu_j(x,t)| \ll 1$, we have $\mu_j(x,t) \sim \cos(x - \omega_j t + \phi_j)$ — we get the ordinary Fourier series!
- If there are no interactions ($N = 1$, i.e., just one wave), we have $\mu(x,t) = \mathrm{cn}^2(x - \omega t + \phi|m)$, which is a Jacobian elliptic function with modulus m (a cnoidal wave).
- The amplitudes of the nonlinear oscillations are given by

$$A_j = \begin{cases} \frac{2}{\lambda}(E_{\mathrm{ref}} - E_{2j}), & \text{for solitons;} \\ \frac{1}{2\lambda}(E_{2j+1} - E_{2j}), & \text{otherwise (radiation).} \end{cases}$$

On Moduli and Wave Numbers

- In the hyperelliptic representation, we have $k_j = 2\pi j/L$ as in Fourier theory (the wave numbers are "commensurable").
- We can compute the elliptic modulus m_j of the hyperelliptic functions, which is called the soliton index, from the discrete spectrum as

$$m_j = \frac{E_{2j+1} - E_{2j}}{E_{2j+1} - E_{2j-1}}, \quad 1 \leq j \leq N.$$

  Then, the oscillations fall into three general categories:
  - $m_j \gtrsim 0.99 \Rightarrow$ solitons [note $\mathrm{cn}^2(x|m_j = 1) = \mathrm{sech}^2(x)$],
  - $m_j \gtrsim 0.5 \Rightarrow$ nonlinearly interacting cnoidal waves, "less nonlinear" than the solitons (e.g., Stokes waves),
  - $m_j \ll 1.0 \Rightarrow$ radiation [note $\mathrm{cn}^2(x|m_j = 0) = \cos^2(x)$].

Numerics of the DST (I)

- For an e-value problem with periodic potential, we can introduce an "appropriate" basis of eigenfuctions and appeal to Floquet's theorem to obtain

$$\mathbf{\Phi}(x + L, E) = \boldsymbol{\alpha}(x, E)\mathbf{\Phi}(x, E),$$

where $\boldsymbol{\alpha}$ is the monodromy matrix and, e.g., $\mathbf{\Phi} = \begin{pmatrix} \phi & \phi_x \\ \phi^* & \phi_x^* \end{pmatrix}$.

- The main spectrum is those E_j s.t. $\Delta(E_j) := \frac{1}{2} \operatorname{tr} \boldsymbol{\alpha}(x, E_j) = \pm 1$.
- The auxiliary spectrum is those μ_j s.t. $\alpha_{21}(x, \mu_j) = 0$.
- **Q:** What is a numerically-accessible quantity that we can us to compute the monodromy matrix and the discrete eigenvalues $\{E_j, \mu_j\}$ of our Schrödinger e-value problem?

Numerics of the DST (II)

- **A:** Rewrite the Schrödinger e-value problem as a first-order system:

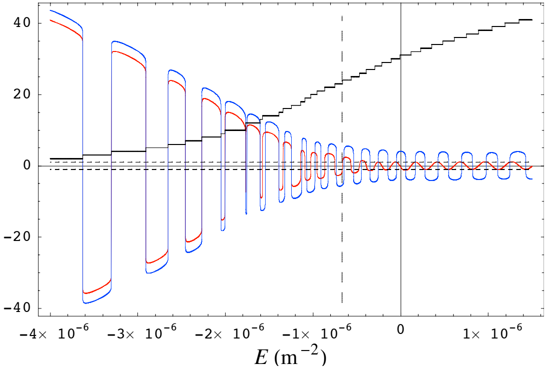
$$\frac{\mathrm{d}}{\mathrm{d}x}\Psi(x,E) = \mathbf{B}(x,E)\Psi(x,E), \quad \mathbf{B}(x,E) = \begin{pmatrix} 0 & 1 \\ -q(x,E) & 0 \end{pmatrix},$$

  where $\Psi := (\psi, \psi_x)^\top$ and $q(x,E) := \lambda\eta(x,0) + E$.
- Integrating we have $\Psi(x,E) = \exp\{\int \mathbf{B}(x,E)\,\mathrm{d}x\}$.
- Now, $\eta(x,0)$ is usually a discrete data set, i.e. its values are only known at $x = x_j := j\Delta x$, $j \in \{0,\dots,M-1\}$.
- Then, we immediately have that $\Psi(x_{j+1},E) = e^{\Delta x \mathbf{B}(x_j,E)}\Psi(x_j,E)$.
- Iterating this relation gives $\Psi(x+L,E) = \mathbf{M}(x,E)\Psi(x,E)$ $\forall x \in [0,L]$, where $\mathbf{M}(x,E) := \prod_{j=M-1}^{0} e^{\Delta x \mathbf{B}(x_j,E)}$ is the so-called scattering matrix.
- Finally, it can be shown that $\frac{1}{2}\operatorname{tr}\boldsymbol{\alpha} \equiv \frac{1}{2}\operatorname{tr}\mathbf{M}$ and $\alpha_{21} \equiv M_{21}$! $\Rightarrow$ We've found a numerically-accessible analogue to the monodromy matrix.

Numerics of the DST (III)

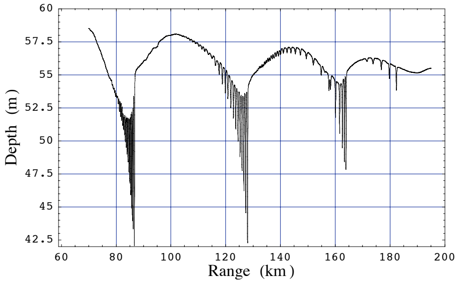
- **Bad news:** finding the ± 1 crossings of the Floquet discriminant $\Delta(E) \equiv \frac{1}{2}\,\mathrm{tr}\,\mathbf{M}$ and the zero crossings of $M_{21}(E)$ is no easy task.
- Osborne 1994 suggests to use "accounting functions" to count the zeros of $M_{11}(E)$, then find the latter's zeros along with those of $M_{21}(E)$ to get a bracketing of the ± 1 crossings of $\Delta(E)$.
- **Problem 1:** The accounting functions suffer numerical instability at the scales the spectrum is located at for the internal soliton model.
- **Problem 2:** Computing all the zeros of $M_{11}(E)$ is a lot of extra work.
- **Idea:** Sample $\Delta(E)$ and $M_{21}(E)$, find brackets for their zero crossings. Then, use the following facts:
  1. $M_{21}(E) = 0$ exactly once between $\pm 1/\pm 1$ crossings of $\Delta(E)$,
  2. $\Delta(E) = 0$ exactly once between $\pm 1/\mp 1$ crossings $\Delta(E)$.
- Finally, keeping the roots bracketed during the root-finding is critical. $\Rightarrow$ Use the regula falsi method; it guarantees the latter and can have superlinear convergence unlike the bisection method (Osborne 1994).

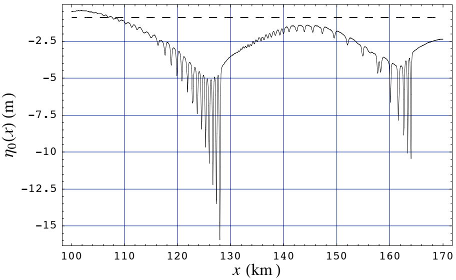
Visualizing the Numerics of the DST



- The zero crossings of $M_{21}(E)$ (the blue oscillations) determine the number of the degrees of freedom in the DST spectrum; each crossing is a μ_j value.
- Consecutive $+1/+1$ and $-1/-1$ crossings of $\Delta(E)$ (the red oscillations) determine the open bands of the DST spectrum; each crossing is an E_j value.

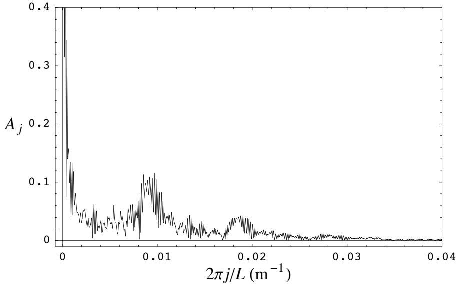
Internal Solitary Waves in the Yellow Sea



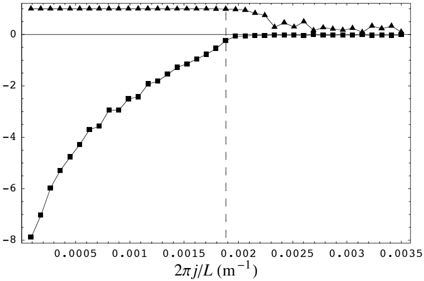


- (⇐) Internal solitary waves in the Yellow Sea from a numerical simulations of solitary-wave generation by bottom topography, using the nonhydrostatic Lamb model, by Chin-Bing et al. (2003). [Depth measured from the sea bottom.]
- (⇒) The two (middle) wave packets we are interested in; part of the $\sigma_t = 22.0$ isopycnal, which is located at a depth of $\approx$ 12m. [Depth measured from the undisplaced isopycnal.]

Fourier Analysis of the Internal Solitary Waves



The FFT spectrum.



The DST spectrum.

- Three things to note about the DST spectrum:
  - the narrower (and distinct) range of wave numbers predicted,
  - the nonlinear non-soliton waves present in the spectrum,
  - there are a lot fewer oscillations modes predicted.
- **Point:** The DST can find solitons in the data set — no guess work necessary.

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