Nonoscillatory Central Schemes on Unstructured Triangulations for Hyperbolic Systems of Conservation Laws

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Minisymposium on Numerical Methods for First-Order PDEs,

8th IMACS International Symposium on
Iterative Methods in Scientific Computation,

College Station, Texas
November 16, 2006

The Big Picture

- Motivation:
  - How should we design "genuinely multidimensional" limiters and (central Godunov-type) schemes on unstructured grids?
  - Need for a simple & fast predictor algorithm for the time-dependent extension of L^1-minimization FEM (J.-L. Guermond & B. Popov).
- Background:
  - High-resolution central schemes: Nessyahu & Tadmor (1992) → Jiang & Tadmor (1998) → Kurganov & Tadmor (2000) → Kurganov, Noelle, Petrova (2001) ...
  - Extension to unstructured grids: Arminjon *et al.* (1997).
  - Semi-discere central-upwind version: Kurganov & Petrova (2005).
- Outline of the talk:
  1. Hyperbolic systems of conservation laws and 2D central schemes.
  2. The minimum-angle-plane reconstruction.
  3. Construction of the staggered grid corresponding to an *unstructured* triangulation.
  4. Numerical results for equations with *convex* and *nonconvex* fluxes.

Statement of the Problem and Notation

- Consider the initial-boundary-value problem for a 2D *hyperbolic* system of conservation laws:

$$\begin{cases} \vec{q}_t + \vec{f}(\vec{q})_x + \vec{g}(\vec{q})_y = 0, & (x, y, t) \in \Omega \times (0, T], \\ \vec{q}(x, y, t = 0) = \vec{q}_0(x, y), & (x, y) \in \Omega, \\ \vec{q}(x, y, t) = \vec{q}_{\mathrm{BC}}(x, y, t), & (x, y, t) \in \partial\Omega \times (0, T]. \end{cases}$$

- $\Omega \subset \mathbb{R}^2$ is the interior of a polygonal domain and $\partial\Omega$ its boundary.
- $\mathcal{T} = \{\tau_i\}$ is a *conforming* triangulation of $\bar{\Omega}$.
- $\bar{w}^n$ is a piecewise-constant approximation to the cell averages of $\vec{q}$ on $\mathcal{T}$ at time t^n.
- $\mathcal{S} = \{\sigma_k\}$ is the *staggered* grid — the "dual" of $\mathcal{T}$.
- $\bar{\mathfrak{w}}^n$ is the analogue of $\bar{w}^n$ on $\mathcal{S}$.

Overview of the 2D Central Scheme

1. Perform a slope-limited piecewise-linear reconstruction on $\mathcal{T}$.

$$\bar{w}^n \longrightarrow w^n$$

2. Evolve the cell averages on the staggered grid $\mathcal{S}$ in time.

$$w^n \longrightarrow \bar{\mathfrak{w}}^n, \quad \bar{\mathfrak{w}}^n \longrightarrow \bar{\mathfrak{w}}^{n+1}$$

3. Project the solution from $\mathcal{S}$ back onto $\mathcal{T}$.

$$\bar{\mathfrak{w}}^{n+1} \longrightarrow \mathfrak{w}^{n+1}, \quad \mathfrak{w}^{n+1} \longrightarrow \bar{w}^{n+1}$$

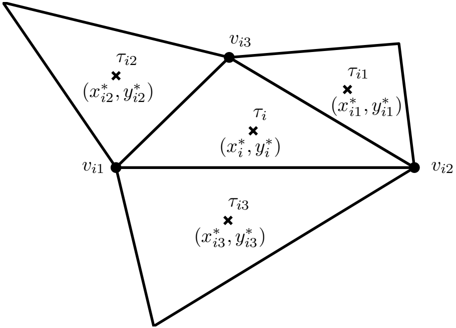
The good, the bad and the ugly:

- No need to solve a Riemann problem at each cell interface!
- Need to define $\mathcal{S}$ in a “reasonable” manner.
- Need to be able to perform a nonoscillatory reconstruction on $\mathcal{S}$.

The Minimum-Angle-Plane Reconstruction

The algorithm:

1. Given an element $\tau_i \in \mathcal{T}$ and its neighbors τ_{ij}, $1 \leq j \leq m$, find all $\binom{m+1}{3}$ possible planes.
2. Find the plane that makes the smallest angle with the horizontal, and use it to find a *limited* gradient.



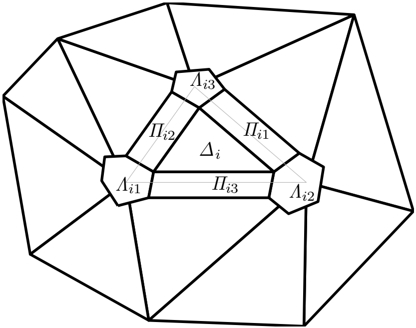
Note that

- This "genuinely 2D" limiter behaves like minmod with a UNO flavor (i.e., $\approx$ Durlofsky–Engquist–Osher but $> 1^{\text{st}}$ order near extrema).
- No particular geometry and/or connectivity is assumed in the design of the limiter, and there are no *ad hoc* parameters.
- Limiter works exactly the same way on $\mathcal{S}$ as on $\mathcal{T}$.

The Staggered Grid

The dual elements:

1. Triangles Δ_i.
2. Polygons Λ_{ij}.
3. Parallelograms Π_{ij}.



The usual CFL condition

$$\Delta t < \tfrac{1}{3} \cdot \min_i |\tau_i| / S_{\max},$$

$S_{\max}$ = fastest wave's speed, is good enough.

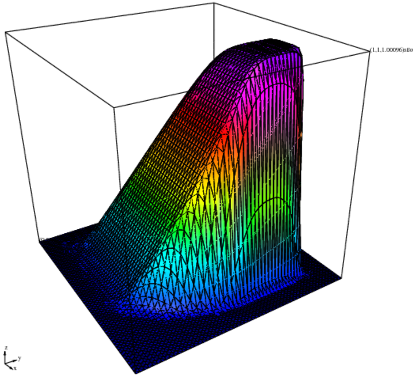
In particular:

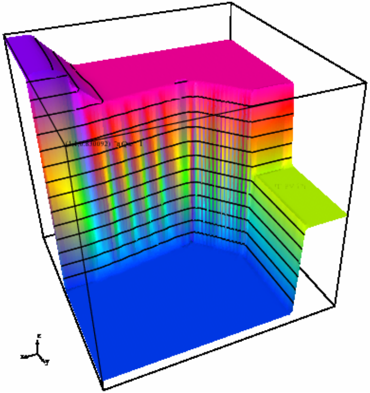
- If $|\Delta_i| = |\Pi_{ij}| = 0$, the staggered grid becomes the Voronoi diagram.
- If the maximum local speed of propagation is used, we get the analogue of Kurganov & Tadmor's modified central differencing for triangulations.

Numerical Results for a *Convex* Flux

- Riemann problems for the 2D inviscid Burgers equation

$$u_t + \left(\tfrac{1}{2}u^2\right)_x + \left(\tfrac{1}{2}u^2\right)_y = 0.$$

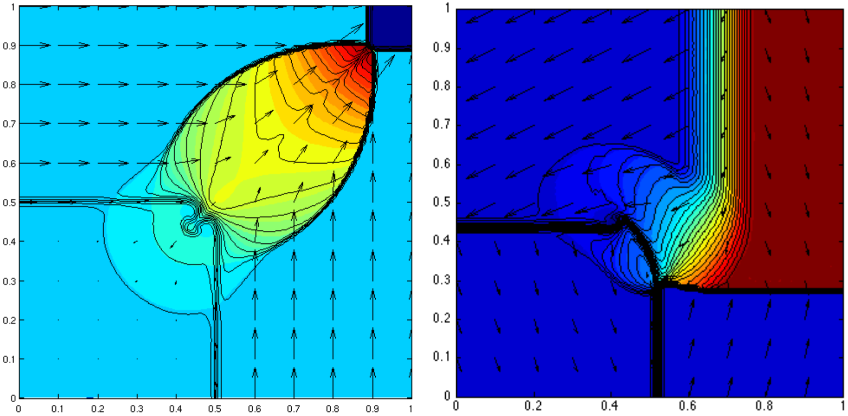




- (L) 6,272 elements and 19,041 dual elements.
- (R) 12,800 elements and 38,721 dual elements.

Riemann Problem for the Euler Equations

- $\vec{q} = (\rho, \rho u, \rho v, E)^\top$, $\vec{f}(\vec{q}) = \left(\rho u, \rho u^2 + p, \rho u v, u(E+p)\right)^\top$,
  $$\vec{g}(\vec{q}) = \left(\rho v, \rho u v, \rho v^2 + p, v(E+p)\right)^\top,$$
- For an ideal gas: $p = (\gamma - 1)\left[E - \frac{1}{2}\rho(u^2 + v^2)\right]$, $\gamma = 1.4$.

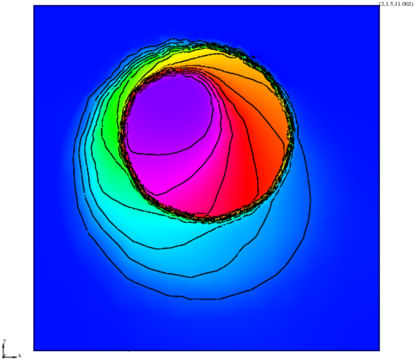


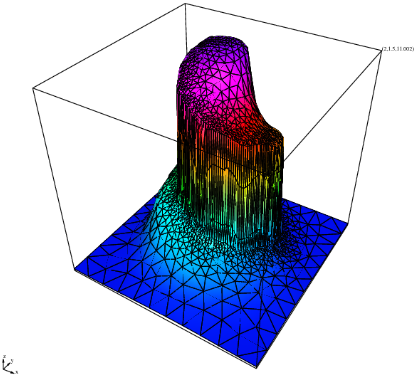
- Very fine mesh ($\min_i \operatorname{diam}(\tau_i) = \sqrt{2}/256$) with CFL $= 0.275$.

Numerical Results for a *Nonconvex* Flux (I)

- Riemann problem for the 2D scalar equation

$$u_t + (\sin u)_x + (\cos u)_y = 0.$$

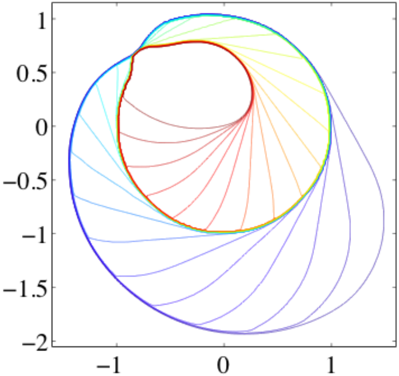


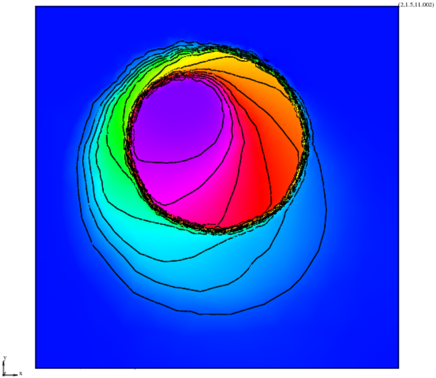


- Adapted mesh with (only) 3,264 elements and 9,837 dual elements, CFL $= \frac{1}{6}$.

Numerical Results for a *Nonconvex* Flux (II)

- Kurganov, Petrova & B. Popov reported that less compressive / higher order limiters (e.g., WENO5, MM2, SB) do *not* resolve the resulting composite wave correctly. The MAPR passes this test!





- (L) 90,000 element Cartesian tensor-product mesh.

Selected Bibliography

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