Thermal shock waves under a Maxwell–Cattaneo model with temperature-dependent conductivity

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The Big Picture

- Motivation:
  - Exact evolution of a thermal shock, under nonlinear hyperbolic (e.g., Maxwell–Cattaneo) models, propagating into a half-space is unknown.
  - In practice, it is useful to have analytical expressions for the shock location and strength as functions of time.

- Outline of the talk:
  1. Mathematical formulation of heat shock wave propagation through a rigid conduction at rest with temperature-dependent conductivity.
  2. Solution of the linearized equations, singular surface theory results.
  3. Rankine–Hugoniot jump conditions for the nonlinear equations, nonlinear shock speed and an *ad-hoc* solution.
  4. Comparison of the linear and *ad-hoc* solutions to the numerical solution of the nonlinear equations.
  5. Conclusions, discussions, open questions and future work.

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Problem formulation and governing equations

- The basic equations are

$$\underbrace{C_0\vartheta_t + q_x = 0}_{\text{conservation of energy}}\ , \quad \underbrace{\lambda_0 q_t + q = -K\vartheta_x}_{\text{MC constitutive law}}\ , \quad \underbrace{K(\vartheta) = K_0[1+\beta(\vartheta-\vartheta_{\mathrm{R}})]}_{\text{temperature-dependent conductivity}},$$

where C_0, λ_0, K_0 are the heat capacity, thermal relaxation time and thermal conductivity at the reference temperature ϑ_{R}.

- We can nondimensionalize and write a quasi-linear hyperbolic system

$$\begin{pmatrix}\theta\\ q\end{pmatrix}_t + \begin{pmatrix}0 & 1\\ c_0^2(1+\epsilon\theta) & 0\end{pmatrix}\begin{pmatrix}\theta\\ q\end{pmatrix}_x = \begin{pmatrix}0\\ -q/\tau_0\end{pmatrix},$$

where $\tau_0 = \frac{K_0\lambda_0}{C_0 L^2}$, $c_0 = \tau_0^{-\frac{1}{2}}$ and $\epsilon = \beta\vartheta_0$.

- Finally, we are interested in the signaling initial-boundary value problem with a quiescent initial state:

$$\theta(0,t) = H(t), \quad t > 0; \quad \theta(x,t) = q(x,0) = 0, \quad x > 0.$$

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Reduction to the damped wave equation

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2. Eliminate q to get the DWE

$$\theta_{tt} + \tau_0^{-1}\theta_t - c_0^2\theta_{xx} = 0.$$

- The initial-boundary conditions are

$$\theta(0,t) = H(t), \quad \theta(\infty,t) = 0, \quad t > 0; \quad \theta(x,0) = \theta_t(x,0) = 0, \quad x > 0.$$

- We can solve this analytically using the Laplace transform (on t) :

$$\theta(x,t) = H(c_0 t - x)\left[e^{-\frac{x}{\varkappa}} + \frac{x}{\varkappa}\int_x^{c_0 t} e^{-\frac{\zeta}{\varkappa}} \frac{I_1\left(\varkappa^{-1}\sqrt{\zeta^2 - x^2}\right)}{\sqrt{\zeta^2 - x^2}}\, d\zeta\right],$$

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Solution by singular surface theory

- So, what does this tells us about how an initial jump (shock) at the boundary evolves as it propagates into the half-space $x > 0$.
- Suppose $\Sigma(t)$ is a singular surface (e.g., a wavefront), then we define

$$[\![\mathfrak{F}]\!](t) \equiv \mathfrak{F}^{-}(t) - \mathfrak{F}^{+}(t), \quad \mathfrak{F}^{\pm} \equiv \lim_{x \to \Sigma(t)^{\pm}} \mathfrak{F}(x,t).$$

- Using Hadamard's lemma (condition of dynamic compatibility) we can derive a Bernoulli-type ODE for $[\![\theta]\!](t)$ and readily obtain that

$$[\![\theta]\!]_{\mathrm{DWE}}(t) = \exp\left(-\frac{t}{2\tau_0}\right), \quad \Sigma_{\mathrm{DWE}}(t) = c_0 t.$$

- Notice that we can also evaluate the integral representation of the exact solution as $x \to (c_0 t)^{\pm}$ to obtain the same result, i.e.,

$$\theta((c_0 t)^{-}, t) = \mathrm{e}^{-\frac{c_0 t}{\varkappa}}, \quad \theta((c_0 t)^{+}, t) = 0, \quad \varkappa = 2c_0\tau_0.$$

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The Rankine–Hugoniot conditions for a nonlinear shock

- The jump or compatibility conditions across a shock wave are

$$[\![\theta]\!]\frac{\mathrm{d}\Sigma}{\mathrm{d}t} = [\![q]\!], \qquad [\![q]\!]\frac{\mathrm{d}\Sigma}{\mathrm{d}t} = [\![c_0^2(\theta + \tfrac{1}{2}\epsilon\theta^2)]\!].$$

- Thanks to $\theta^+(t) \equiv 0$, we can solve for the shock speed:

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- Assuming $[\![\theta]\!](t) \approx [\![\theta]\!]_{\mathrm{DWE}}(t) = \exp(-\frac{t}{2\tau_0})$, we can do the integral:

$$\Sigma(t) \approx c_0 t + 4c_0\tau_0\left[\mathfrak{S}(0) - \mathfrak{S}(t) + \ln\left(\frac{1 + \mathfrak{S}(t)}{1 + \mathfrak{S}(0)}\right)\right],$$

where $\mathfrak{S}(t) = \sqrt{1 + \frac{1}{2}\epsilon\exp(-\frac{t}{2\tau_0})}$.

- Notice that $\lim_{\epsilon\to 0}\mathfrak{S}(t) = 1$, so we recover the linear-theory expression for $\Sigma(t)$ in the appropriate limit.

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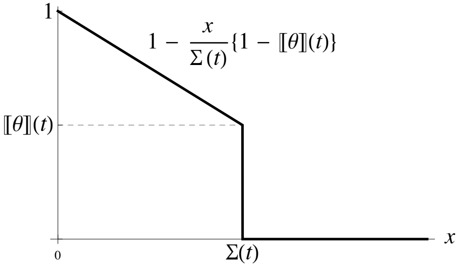
An *ad-hoc* solution of the nonlinear problem

- For moderate values of τ_0, a linear interpolation between the boundary value and the jump at the wavefront is quite accurate:

$$\theta(x,t) \approx H\left(1 - \frac{x}{\Sigma(t)}\right)\left(1 - \frac{x}{\Sigma(t)}\left\{1 - [\![\theta]\!](t)\right\}\right).$$



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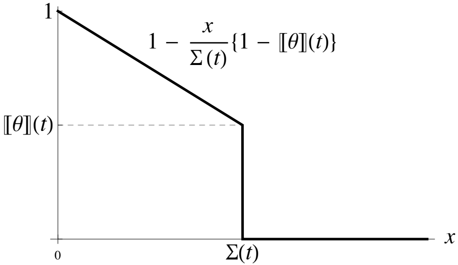

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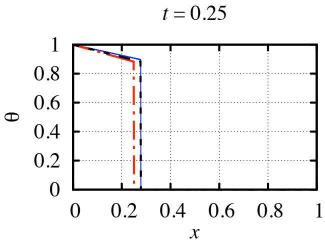
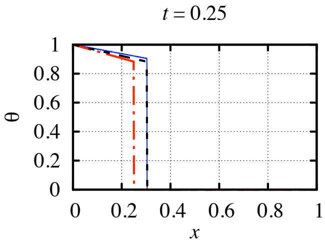
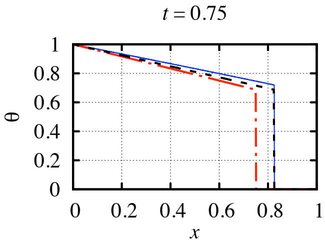
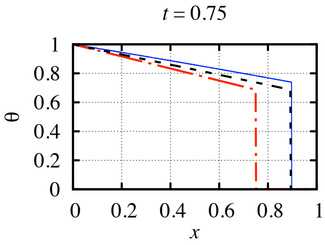
Comparison of analytical and numerical solutions









$\epsilon = 0.5$ $\epsilon = 1.0$

Summary & conclusions

- Proposed a simple physically-motivated method for constructing an approximate analytical solution for the propagation of a thermal shock in a conductor with temperature-dependent conductivity under the Maxwell–Catteneo theory.
- Benchmarked the *ad-hoc* solution against direct numerical simulations, finding excellent agreement even for $\epsilon, t = \mathcal{O}(1)$.
- The shock location was approximated with great accuracy but the jump is still that of the linear theory; is there a way to improve this aspect of the *ad-hoc* solution? Introduce a lag? Semi-linearize?
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- Benchmarked the *ad-hoc* solution against direct numerical simulations, finding excellent agreement even for $\epsilon, t = \mathcal{O}(1)$.
- The shock location was approximated with great accuracy but the jump is still that of the linear theory; is there a way to improve this aspect of the *ad-hoc* solution? Introduce a lag? Semi-linearize?
- The approach presented is very general applicable to a myriad of sock phenomena governed by quasi-linear hyperbolic PDEs.

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