Interactions of dissipative solitons in a weakly-nonlinear acoustic Euler-α model

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The big picture

- Motivation:
  - Effect of fluctuations upon mean flow of a compressible fluid? Average to obtain a coarse-grained eq. (RANS & LES in turbulence).
  - Averaging in the variational principle leads to "nonlinearly dispersive" corrections.
  - Vanishing-viscosity $\rightarrow$ vanishing-dispersion limit from NS to Euler?
  - Implications of a such model for acoustics? (Canonical problems.)

- Outline of the talk:
  1. Derivation of weakly-nonlinear Lagrangian-averaged Euler-α (LAE-α) model.
  2. "Dispersive Taylor shocks" under the linearized eq. of motion (EoM).
  3. Energy analysis and kink-shaped traveling wave solutions (TWSs) of the nonlinear problem.
  4. Numerical simulations of soliton and anti-soliton interactions using scheme with internal iterations.

Background

- α-models of turbulence (Foias, Holm, Titi, Guermond, et al., ca. 1999–2003).
  - Introduce energy penalty to prevent creation of excitations below a minimal length scale, α.
- For compressible fluids, α is the flow-dependent small-scale amplitude fluctuations over whose phase the Lagrangian mean is taken.
- α can also represent:
  - the cell size in a finite-scale Lagrangian numerical simulation (Margolin, *TRSA*, 2009),
  - the first material modulus of a second-grade non-Newtonian fluids (formal analogy).
- Bhat & Fetecau (*DCDS-B*, 2006): LAE-α model of compressible flow for perfect gases (i.e., polytropic eq. of state).
- Lesser & Seebass (*JFM*, 1968): finite-amplitude, aka weakly-nonlinear, approximation under barotropic eq. of state.

Eqs. of motion for 1D homentropic flow of an inviscid fluid

- Mass, momentum and energy conservation:

$$\varrho_t + (\varrho u)_x = 0,$$
$$\varrho(u_t + uu_x) + \wp_x = 0\alpha^2[\varrho_x(u_{xt} + uu_{xx}) + \varrho(u_x u_{xx} + u_{xxt} + uu_{xxx})],$$
$$\eta_t = \eta_t + u\eta_x = 0,$$

where $\mathbf{u} = \nabla\phi = \big(u(x,t), 0, 0\big)$.

- Close system w/ quadratic approximation to barotropic eq. of state

$$\wp - \wp_{\mathrm{e}} = \varrho_{\mathrm{e}} c_{\mathrm{e}}^2 \left[\frac{\varrho-\varrho_{\mathrm{e}}}{\varrho_{\mathrm{e}}} + (\beta - 1)\left(\frac{\varrho-\varrho_{\mathrm{e}}}{\varrho_{\mathrm{e}}}\right)^2\right],$$

- Introduce dimensionless variables/quantities

$$s = (\varrho - \varrho_{\mathrm{e}})/\varrho_{\mathrm{e}}, \quad u^\diamond = u/V, \quad \phi^\diamond = \phi/(LV),$$
$$x^\diamond = x/L, \quad t^\diamond = t(c_{\mathrm{e}}/L), \quad \underbrace{\epsilon = V/c_{\mathrm{e}}}_{\text{Mach number}}, \quad \underbrace{a = \alpha/L}_{\text{Knudsen number}}.$$

Weakly-nonlinear theory and reduction to a scalar equation

- To justify continuum equation with averaged fluctuations, $a \ll 1$ (small Knudsen number).
- Assume weak nonlinearity $\epsilon \ll 1$ and take $a^2 = \mathcal{O}(\epsilon)$.
- After some work...

$$\phi_{xx} - \phi_{tt} + a^2 \phi_{ttxx} = \epsilon \left[2(\beta - 1)\phi_t \phi_{xx} + \partial_t (\phi_x)^2 \right],$$

  a new weakly-nonlinear dispersive acoustic equation.
- Interesting limits:
  - $a \to 0$: recover straightforward weakly-nonlinear acoustic wave (or "inviscid Blackstock–Crighton–Lesser–Seebass") equation of I.C.C., C.I.C. & Jordan (*QJMAM*, 2007).
  - $\epsilon \to 0$: recover Love's equation of classical elasticity or van Wijngaarden equation for sound waves in bubbly liquids.

Linearization: Love's equation

1. Set $\epsilon = 0$ to get $\phi_{xx} - \phi_{tt} + a^2\phi_{ttxx} = 0$.
2. The pressure satisfies the same eq., consider the signaling IBVP:

$$\begin{aligned} &p_{xx} - p_{tt} + a^2 p_{ttxx} = 0, && (x,t) \in (0,\infty)\times(0,\infty), \\ &p(0,t) = H(t), \quad p(\infty,t) = 0, && t \in (0,\infty), \\ &p(x,0) = 0, \quad p_t(x,0) = 0, && x \in (0,\infty). \end{aligned}$$

3. Solve by Laplace transform in t to get
$$p(x,t) = H(t)\left[1 - \frac{2}{\pi}\int_0^{1/a} \frac{\cos(\eta t)}{\eta} \sin\left(\frac{\eta x}{\sqrt{1-a^2\eta^2}}\right) \mathrm{d}\eta\right].$$
4. Or, solve by Fourier sine transform in x to get
$$p(x,t) = H(t)\left[1 - \frac{2}{\pi}\int_0^{\infty} \frac{\sin(\varsigma x)}{\varsigma(1+a^2\varsigma^2)} \cos\left(\frac{\varsigma t}{\sqrt{1+a^2\varsigma^2}}\right) \mathrm{d}\varsigma\right].$$
5. Notice $\eta = \varsigma/\sqrt{1+a^2\varsigma^2}$, similar to Stokes' first problem for second-grade fluid (I.C.C. & C.I.C., *Acta Mech.*, 2010).

Solution to the signaling IBVP for Love's equation

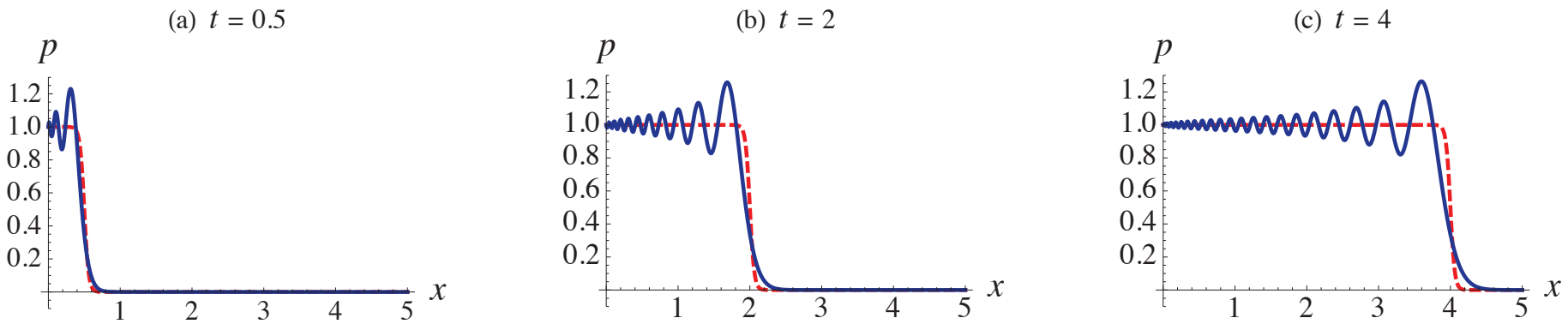


Figure: The solution profile p vs. x of IBVP (solid) compared to the Taylor shock profile $\frac{1}{2}\{1 - \tanh[(x-t)/(2a)]\}$ (dashed); $a = 0.03$.

- Similar to strongly-dispersive regime of a nonlinear strain model of the growth of long bones proposed (Porubov & Maugin, *IJNLM*, 2011).
- Also reminiscent of zero-dispersion limit of KdV equation (e.g., Grava & Klein, *CPAM*, 2007).

Energy analysis

- Assume a localized solitary wave $\phi_x, \phi_t \to 0$ as $|x| \to \infty$, then integrate EoM over $x \in (-\infty, +\infty)$:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -2\epsilon(\beta - 3/2)\int_{-\infty}^{+\infty} (\phi_t)^2 \phi_{xx}\,\mathrm{d}x.$$

- Energy is given by

$$\mathcal{E} := \frac{1}{2}\int_{-\infty}^{+\infty} \underbrace{(\phi_t)^2}_{\text{kinetic energy}} + \underbrace{(\phi_x)^2}_{\text{potential energy}} + \underbrace{a^2(\phi_{xt})^2}_{\text{rotational kinetic energy at microscale}} \mathrm{d}x.$$

- Sign of $\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}$ is not definite; for $\beta = \frac{3}{2}$, $\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = 0$.
- For $\beta \neq \frac{3}{2}$, $\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = 0$ only for certain distinguished solutions, i.e., dissipative solitons (as in C.I.C. & Velarde, *Physica D*, 1995).
- Non-conservative nature of EoM entirely due to the nonlinearity. Linearized version ($\epsilon = 0$) is obviously conservative.

Traveling wave solutions

- EoM invariant under $x \mapsto -x$, consider only right-traveling waves: $\phi(x,t) = F(\xi)$ with $\xi := x - ct$.
- EoM is reduced to an ODE for $f = F'$:

$$3a^2c^2(f')^2 + 3(1-c^2)f^2 + 2\epsilon\beta c f^3 = 6\mathfrak{K}_1 f + \mathfrak{K}_2.$$

- For supersonic wave ($c > 1$) with $\mathfrak{K}_1 = \mathfrak{K}_2 = 0$:

$$f(\xi) = A_0 \operatorname{sech}^2\left[\frac{(\xi-\xi_0)\sqrt{c^2-1}}{2ac}\right], \qquad A_0 = \frac{3(c^2-1)}{2c\epsilon\beta},$$

- In terms of the potential ϕ:

$$F(\xi) = \frac{2acA_0}{\sqrt{c^2-1}} \tanh\left[\frac{(\xi-\xi_0)\sqrt{c^2-1}}{2ac}\right],$$

which is a topological soliton or kink.

Finite-difference scheme with internal iterations

- Introduce auxiliary function $\psi := \phi_t$, EoM becomes

$$\phi_{tt} = \phi_{xx} + a^2\phi_{ttxx} - \epsilon[2(\beta-1)\psi\phi_{xx} + 2\phi_x\psi_x].$$

- Use three-time-level, linearly-implicit discretization:

$$\delta_{t+}\delta_{t-}\Phi_j^n = \delta_{x+}\delta_{x-}\left[\tfrac{1}{4}(\Phi_j^{n+1} + 2\Phi_j^n + \Phi_j^{n-1})\right] + a^2\delta_{t+}\delta_{t-}\delta_{x+}\delta_{x-}\Phi_j^n - \epsilon\left[2(\beta-1)\Psi_j^n\delta_{x+}\delta_{x-}\Phi_j^n + 2\delta_{x0}\Phi_j^n\delta_{x0}\Psi_j^n\right],$$

where $\Phi_j^n \approx \phi(x_j, t^n)$ and $\Psi_j^n \approx \psi(x_j, t^n)$.

- Internal iterations (e.g., C.I.C. & Velarde, *IJBC*, 1994) on k, guess $\Psi_j^{n,0} = \delta_{t-}\Phi_j^n$, find Φ_j^{n+1} from difference eq., then $\Psi_j^{n,k+1} = \delta_{t0}\Phi_j^n$.
- Convergence criterion $\max_j |\Psi_j^{n,k+1} - \Psi_j^{n,k}| < 10^{-8} \max_j |\Psi_j^{n,k}|$ is met within 3 to 10 iterations.

Simulation of soliton–anti-soliton interaction, $\beta < 3/2$

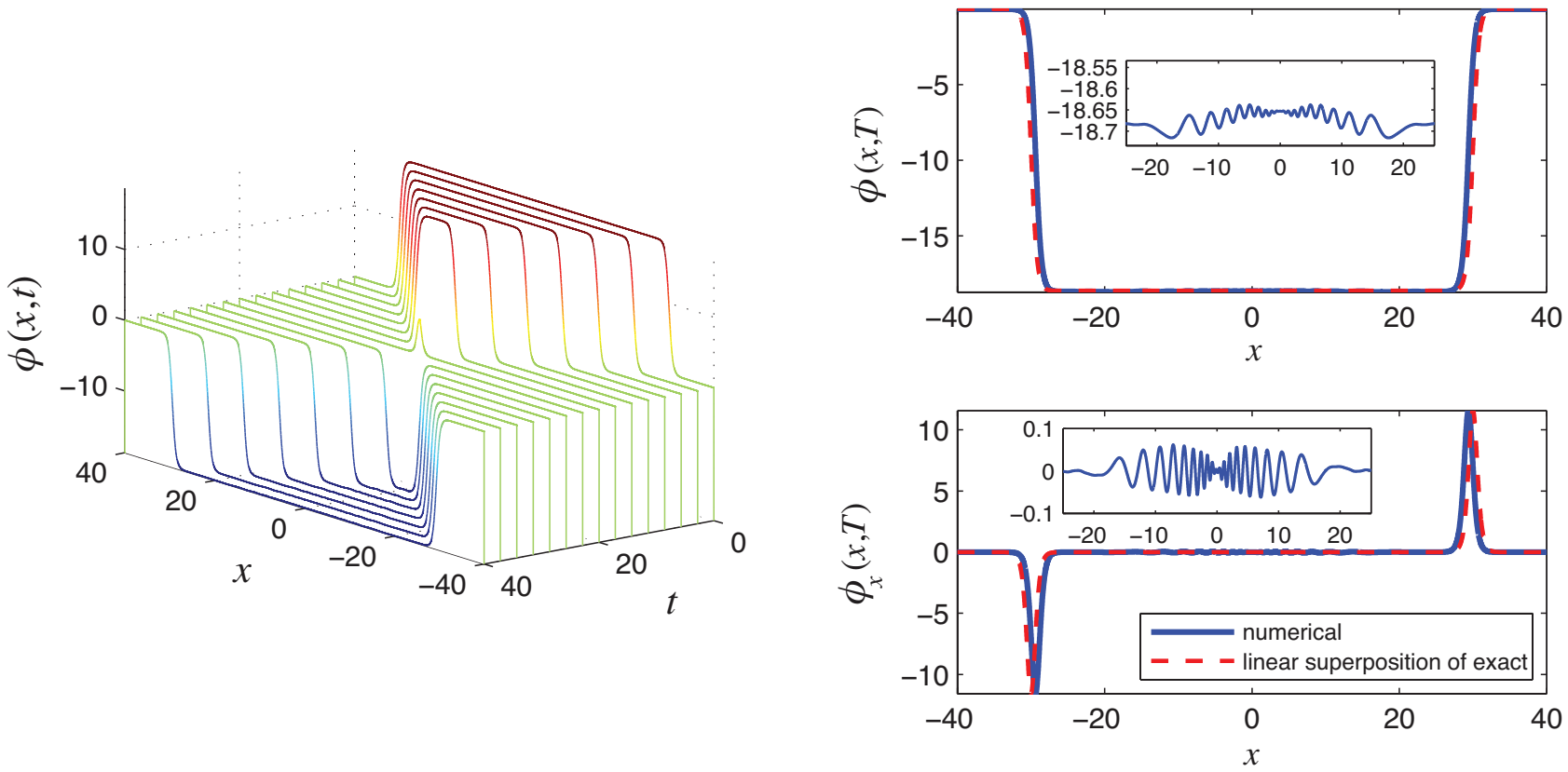


Figure: $\beta = 1.2$, $c_{1,2} = \pm 1.5$, $\xi_{0,\{1,2\}} = \mp 30$ and $T = 40$: space-time plot (left) and comparison of the post-interaction solitons with a linear superposition of the exact shapes (right). Clearly, the two solitons collide and re-emerge, retaining their shape ("identity"), save for a phase shift and the emission of small-amplitude radiation.

Simulation of soliton–soliton interaction, $\beta < 3/2$

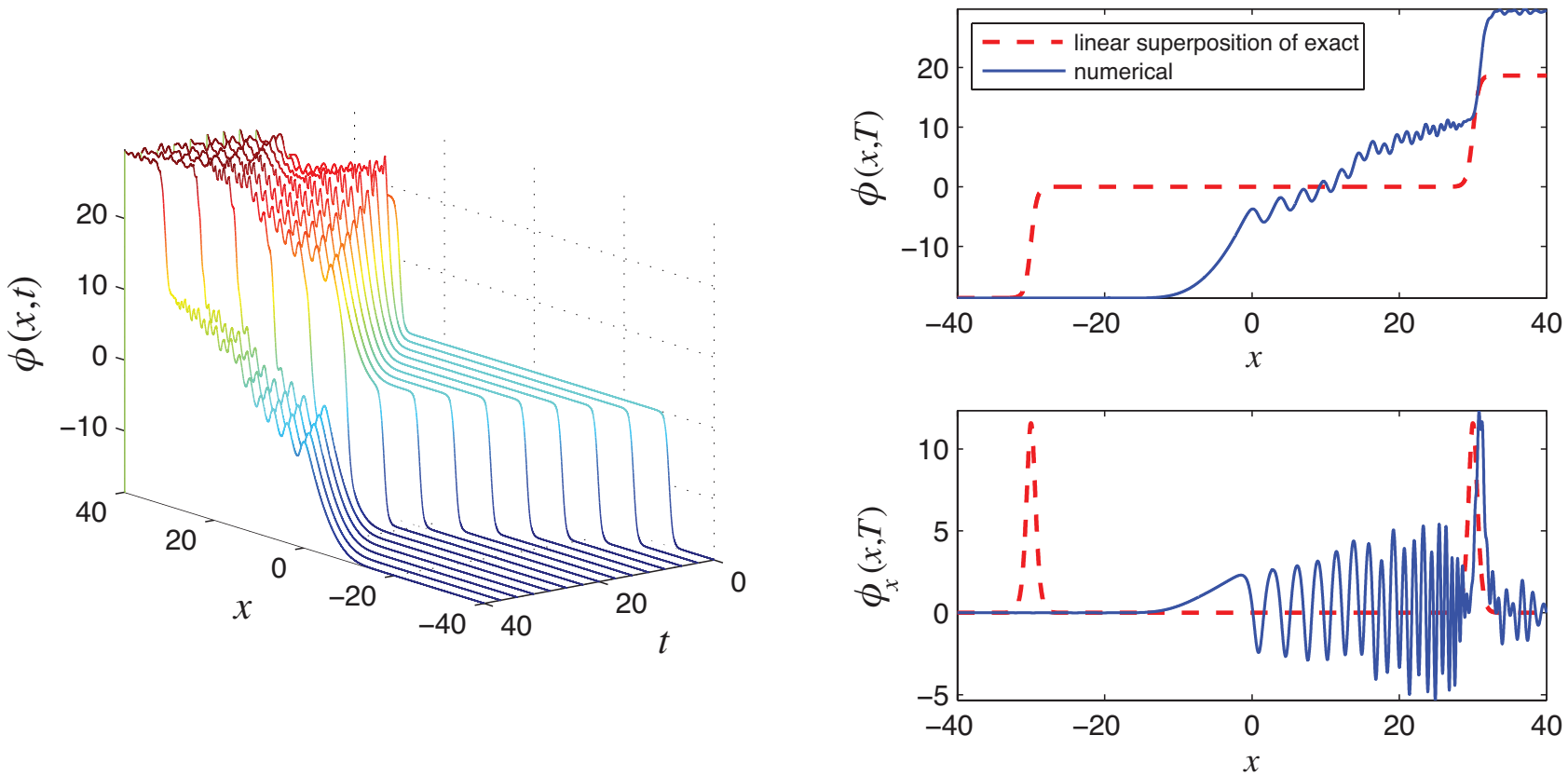
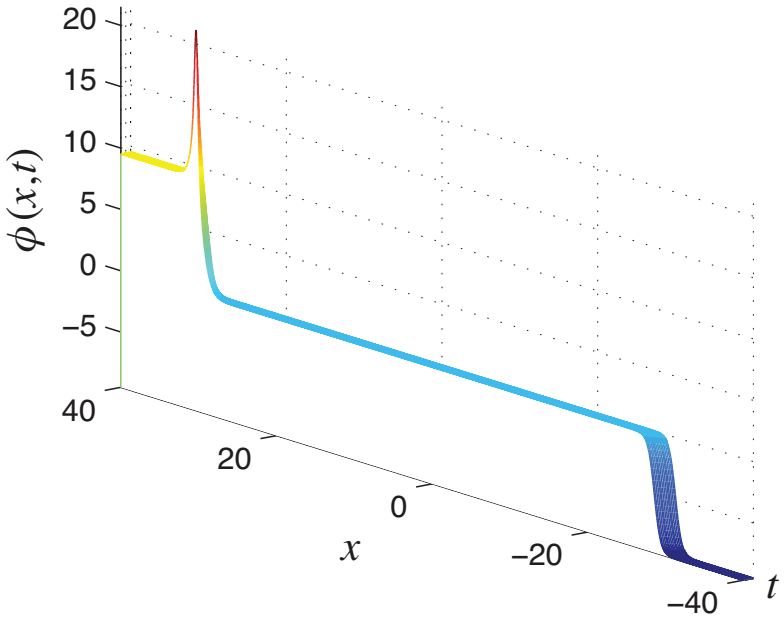


Figure: $\beta = 1.2$, $c_{1,2} = \pm 1.5$, $\xi_{0,\{1,2\}} = \mp 30$ and $T = 40$: space-time plot (left) and comparison of the post-interaction solitons with a linear superposition of the exact shapes (right). The right soliton disintegrates almost immediately, and so does the left one upon reaching the remains of the right soliton.

Simulation of soliton–soliton interaction, $\beta > 3/2$



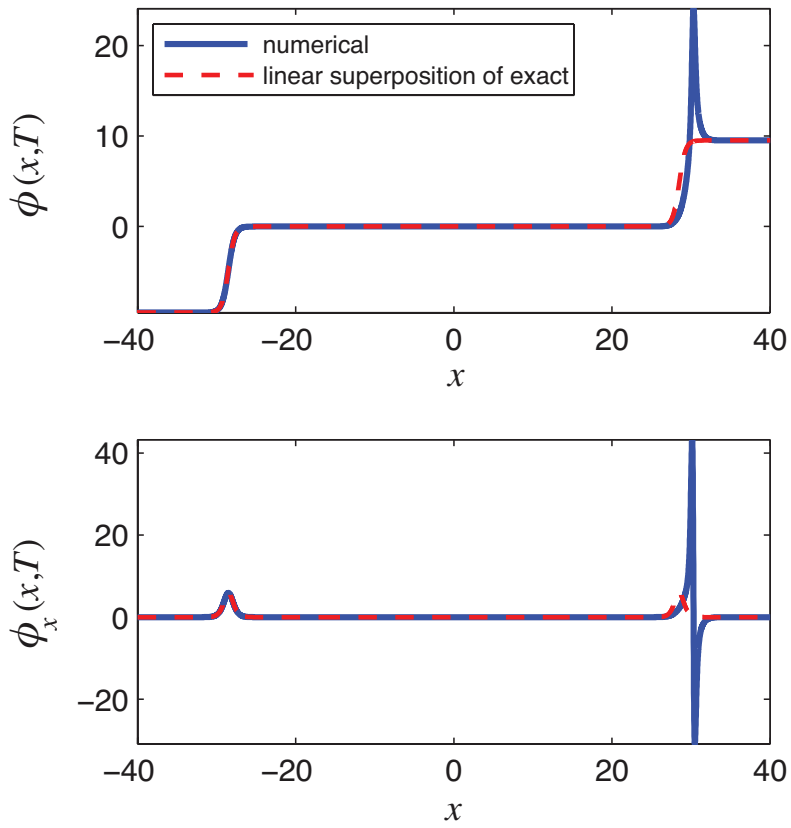


Figure: $\beta = 2.35$, $c_{1,2} = \pm 1.5$, $\xi_{0,\{1,2\}} = \mp 30$ and $T = 1$: space-time plot (left) and comparison of the post-interaction solitons with a linear superposition of the exact shapes (right). The right soliton develops a "horn" that grows without bound.

Summary & conclusions

- What we have done:
  1. New 1D weakly-nonlinear LAE-α model eq. valid for gases & compressible liquids.
  2. Kinks TWSs can retain their identity after a collision.
  3. These TWS are dissipative solitons.
  4. Unstable combinations of kinks that disintegrate even before impact.
  5. Instability can be explosive.
- What we have yet to do:
  1. Investigate of the stability of the kink solutions.
  2. Check whether $\exists$ ranges of the wave speed c for which the soliton–anti-soliton interaction leads to a bound state (like ϕ^4).
  3. Study the other weakly-nonlinear models (Rassmusen, Sørensen, et al., 2008–2011). Can also support dissipative solitons?

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