Interactions of dissipative solitons in a weakly-nonlinear acoustic Euler-$\alpha$ model

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Recent trends in nonlinear wave phenomena—Achievements and challenges: A symposium dedicated to Prof. Christo I. Christov on the occasion of his 60th birthday

7th IMACS International Conference on Nonlinear Evolution Equations and Wave Phenomena: Computation and Theory

Athens, Georgia, April 5, 2011

*A Travel Award (through NSF grant DMS-1048816) is kindly acknowledged.
The big picture

- Motivation:
  - Effect of fluctuations upon mean flow of a compressible fluid? Average to obtain a coarse-grained eq. (RANS & LES in turbulence).
  - Averaging in the variational principle leads to “nonlinearly dispersive” corrections.
  - Vanishing-viscosity $\rightarrow$ vanishing-dispersion limit from NS to Euler?
  - Implications of a such model for acoustics? (Canonical problems.)

- Outline of the talk:
  1. Derivation of weakly-nonlinear Lagrangian-averaged Euler-$\alpha$ (LAE-$\alpha$) model.
  2. “Dispersive Taylor shocks” under the linearized eq. of motion (EoM).
  3. Energy analysis and kink-shaped traveling wave solutions (TWSs) of the nonlinear problem.
Background

- **α-models** of turbulence (Foias, Holm, Titi, Guermond, et al., ca. 1999–2003).
  - Introduce **energy penalty** to prevent creation of excitations below a minimal length scale, $\alpha$.

- For compressible fluids, $\alpha$ is the flow-dependent **small-scale amplitude fluctuations** over whose phase the Lagrangian mean is taken.

- $\alpha$ can also represent:
  - the cell size in a **finite-scale** Lagrangian numerical simulation (Margolin, *TRSA*, 2009),
  - the first material modulus of a **second-grade** non-Newtonian fluids (formal analogy).

- Bhat & Fetecau (*DCDS-B*, 2006): LAE-$\alpha$ model of compressible flow for perfect gases (i.e., polytropic eq. of state).

Governing equations

Eqs. of motion for 1D homentropic flow of an inviscid fluid

- Mass, momentum and energy conservation:

\[ \rho_t + (\rho u)_x = 0, \]
\[ \rho(u_t + uu_x) + \rho_x = 0\alpha^2[\rho_x(u_{xt} + uu_{xx}) + \rho(u_x u_{xx} + u_{xxt} + uu_{xxx})], \]
\[ \eta_t = \eta_t + u\eta_x = 0, \]

where \( u = \nabla \phi = (u(x, t), 0, 0). \)

- Close system w/ quadratic approximation to barotropic eq. of state

\[ \rho - \rho_e = \rho_e c_e^2 \left[ \frac{\rho - \rho_e}{\rho_e} + (\beta - 1) \left( \frac{\rho - \rho_e}{\rho_e} \right)^2 \right], \]

- Introduce dimensionless variables/quantities

\[ s = (\rho - \rho_e)/\rho_e, \quad u^\diamond = u/V, \quad \phi^\diamond = \phi/(LV), \]
\[ x^\diamond = x/L, \quad t^\diamond = t(c_e/L), \quad \epsilon = V/c_e, \quad a = \alpha/L. \]

Mach number \quad Knudsen number
Weakly-nonlinear theory and reduction to a scalar equation

- To justify continuum equation with averaged fluctuations, $a \ll 1$ (small Knudsen number).

- Assume weak nonlinearity $\epsilon \ll 1$ and take $a^2 = O(\epsilon)$.

- After some work...

$$\phi_{xx} - \phi_{tt} + a^2 \phi_{txx} = \epsilon \left[ 2(\beta - 1)\phi_t \phi_{xx} + \partial_t (\phi_x)^2 \right],$$

a new weakly-nonlinear dispersive acoustic equation.

- Interesting limits:
  - $a \to 0$: recover straightforward weakly-nonlinear acoustic wave (or “inviscid Blackstock–Crighton–Lesser–Seebass”) equation of I.C.C., C.I.C. & Jordan (QJMAM, 2007).
  - $\epsilon \to 0$: recover Love’s equation of classical elasticity or van Wijngaarden equation for sound waves in bubbly liquids.
**Linearization: Love's equation**

1. Set $\epsilon = 0$ to get $\phi_{xx} - \phi_{tt} + a^2 \phi_{ttxx} = 0$.

2. The pressure satisfies the same eq., consider the signaling IBVP:

   
   \[
   p_{xx} - p_{tt} + a^2 p_{ttxx} = 0, \quad (x, t) \in (0, \infty) \times (0, \infty),
   \]

   \[
   p(0, t) = H(t), \quad p(\infty, t) = 0, \quad t \in (0, \infty),
   \]

   \[
   p(x, 0) = 0, \quad p_t(x, 0) = 0, \quad x \in (0, \infty).
   \]

3. Solve by **Laplace transform** in $t$ to get

   \[
   p(x, t) = H(t) \left[ 1 - \frac{2}{\pi} \int_0^{1/a} \frac{\cos(\eta t)}{\eta} \sin \left( \frac{\eta x}{\sqrt{1-a^2\eta^2}} \right) d\eta \right].
   \]

4. Or, solve by **Fourier sine transform** in $x$ to get

   \[
   p(x, t) = H(t) \left[ 1 - \frac{2}{\pi} \int_0^\infty \frac{\sin(\varsigma x)}{\varsigma(1+a^2\varsigma^2)} \cos \left( \frac{\varsigma t}{\sqrt{1+a^2\varsigma^2}} \right) d\varsigma \right].
   \]

Solution to the signaling IBVP for Love’s equation

Figure: The solution profile $p$ vs. $x$ of IBVP (solid) compared to the Taylor shock profile $\frac{1}{2}\{1 - \tanh[(x - t)/(2a)]\}$ (dashed); $a = 0.03$.

- Similar to strongly-dispersive regime of a nonlinear strain model of the growth of long bones proposed (Porubov & Maugin, *IJNLM*, 2011).
- Also reminiscent of zero-dispersion limit of KdV equation (e.g., Grava & Klein, *CPAM*, 2007).
Energy analysis

- Assume a localized solitary wave $\phi_x, \phi_t \to 0$ as $|x| \to \infty$, then integrate EoM over $x \in (-\infty, +\infty)$:

$$\frac{dE}{dt} = -2\epsilon(\beta - 3/2) \int_{-\infty}^{+\infty} (\phi_t)^2 \phi_{xx} \, dx.$$ 

- Energy is given by

$$E := \frac{1}{2} \left[ \int_{-\infty}^{+\infty} (\phi_t)^2 + (\phi_x)^2 + a^2(\phi_{xt})^2 \right] \, dx.$$ 

- Sign of $\frac{dE}{dt}$ is not definite; for $\beta = \frac{3}{2}$, $\frac{dE}{dt} = 0$.

- For $\beta \neq \frac{3}{2}$, $\frac{dE}{dt} = 0$ only for certain distinguished solutions, i.e., dissipative solitons (as in C.I.C. & Velarde, *Physica D*, 1995).

- Non-conservative nature of EoM entirely due to the nonlinearity. Linearized version ($\epsilon = 0$) is obviously conservative.
Traveling wave solutions

- EoM invariant under $x \mapsto -x$, consider only right-traveling waves: $\phi(x, t) = F(\xi)$ with $\xi := x - ct$.

- EoM is reduced to an ODE for $f = F'$:

$$3a^2c^2(f')^2 + 3(1 - c^2)f^2 + 2\epsilon\beta cf^3 = 6\mathcal{R}_1 f + \mathcal{R}_2.$$ 

- For supersonic wave ($c > 1$) with $\mathcal{R}_1 = \mathcal{R}_2 = 0$:

$$f(\xi) = A_0 \text{sech}^2 \left[ \frac{(\xi - \xi_0)\sqrt{c^2 - 1}}{2ac} \right], \quad A_0 = \frac{3(c^2 - 1)}{2c\epsilon\beta},$$

- In terms of the potential $\phi$:

$$F(\xi) = \frac{2acA_0}{\sqrt{c^2 - 1}} \tanh \left[ \frac{(\xi - \xi_0)\sqrt{c^2 - 1}}{2ac} \right],$$

which is a topological soliton or kink.
Finite-difference scheme with internal iterations

- Introduce auxiliary function $\psi := \phi_t$, EoM becomes

$$\phi_{tt} = \phi_{xx} + a^2 \phi_{ttxx} - \epsilon [2(\beta - 1) \psi \phi_{xx} + 2\phi_x \psi_x].$$

- Use three-time-level, linearly-implicit discretization:

$$\delta_t + \delta_t - \Phi^n_j = \delta_x + \delta_x - \left[ \frac{1}{4} (\Phi^{n+1}_j + 2\Phi^n_j + \Phi^{n-1}_j) \right] + a^2 \delta_t + \delta_t - \delta_x + \delta_x - \Phi^n_j$$

$$- \epsilon \left[ 2(\beta - 1) \psi^n_j \delta_x + \delta_x - \Phi^n_j + 2\delta_x0 \Phi^n_j \delta_x0 \psi^n_j \right],$$

where $\Phi^n_j \approx \phi(x_j, t^n)$ and $\psi^n_j \approx \psi(x_j, t^n)$.

- **Internal iterations** (e.g., C.I.C. & Velarde, *IJBC*, 1994) on $k$, guess

$\psi^{n,0}_j = \delta_t - \Phi^n_j$, find $\Phi^{n+1}_j$ from difference eq., then $\psi^{n,k+1}_j = \delta_{t0} \Phi^n_j$.

- Convergence criterion $\max_j |\psi^{n,k+1}_j - \psi^{n,k}_j| < 10^{-8} \max_j |\psi^{n,k}_j|$ is met within 3 to 10 iterations.
Simulation of soliton–anti-soliton interaction, $\beta < 3/2$

**Figure:** $\beta = 1.2$, $c_{1,2} = \pm 1.5$, $\xi_0,\{1,2\} = \mp 30$ and $T = 40$: space-time plot (left) and comparison of the post-interaction solitons with a linear superposition of the exact shapes (right). Clearly, the two solitons collide and re-emerge, retaining their shape ("identity"), save for a phase shift and the emission of small-amplitude radiation.

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Dissipative acoustic solitons  
IMACS Waves 2011: SS #2
Simulation of soliton–soliton interaction, $\beta < 3/2$

**Figure**: $\beta = 1.2$, $c_{1,2} = \pm 1.5$, $\xi_{0,\{1,2\}} = \mp 30$ and $T = 40$: space-time plot (left) and comparison of the post-interaction solitons with a linear superposition of the exact shapes (right). The right soliton disintegrates almost immediately, and so does the left one upon reaching the remains of the right soliton.
Simulation of soliton–soliton interaction, $\beta > 3/2$

**Figure:** $\beta = 2.35$, $c_{1,2} = \pm 1.5$, $\xi_0,\{1,2\} = \mp 30$ and $T = 1$: space-time plot (left) and comparison of the post-interaction solitons with a linear superposition of the exact shapes (right). The right soliton develops a “horn” that grows without bound.
Summary & conclusions

- What we have done:
  1. New 1D weakly-nonlinear LAE-\(\alpha\) model eq. valid for gases & compressible liquids.
  2. Kinks TWSs can retain their identity after a collision.
  3. These TWS are dissipative solitons.
  4. Unstable combinations of kinks that disintegrate even before impact.
  5. Instability can be explosive.

- What we have yet to do:
  1. Investigate of the stability of the kink solutions.
  2. Check whether \(\exists\) ranges of the wave speed \(c\) for which the soliton–anti-soliton interaction leads to a bound state (like \(\phi^4\)).
  3. Study the other weakly-nonlinear models (Rassmusen, Sørensen, et al., 2008–2011). Can also support dissipative solitons?
Selected Bibliography

C. I. Christov & M. G. Velarde, “Dissipative solitons,”

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*Wave Motion* (2011) submitted.