On a Hierarchy of Nonlinearly Dispersive Generalized KdV Equations

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Recent Developments in Continuum Mechanics and Partial Differential Equations
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April 19, 2015
Introduction: Continua with an inherent length scale

- Balance laws of (local) continuum mechanics:
  \[
  \dot{\rho} + \rho \nabla \cdot \mathbf{u} = 0, \quad \rho \mathbf{u} = \nabla \cdot \mathbf{T} + \mathbf{b}.
  \]
  - conservation of mass
  - conservation of linear momentum

- “RRG” theory: let \( \mathbf{T} = \mathbf{T}^{(1)} + \mathbf{T}^{(2)} \), where
  \[
  \mathbf{T}^{(1)} = -\rho \mathbf{I} + 2\mu \text{symm}[\nabla \mathbf{u}] + (\mu_B - \frac{2}{3}\mu)(\nabla \cdot \mathbf{u})\mathbf{I},
  \]
  \[
  \mathbf{T}^{(2)} = \rho \Psi \left\{ 2 \text{symm}[\nabla \mathbf{u}] + 2(\nabla \mathbf{u})^\top \nabla \mathbf{u} - 4(\text{symm}[\nabla \mathbf{u}])^2 \right\}.
  \]
  (Rubin et. al, J. Appl. Phys., 1995)

- \( \Psi \geq 0 \), units of \( L^2 \), is the dispersion function; can depend only on the invariant \( (\text{symm}[\text{grad} \mathbf{u}])^2 \).

- (Connections to “second-grade fluid” of Coleman & Gurtin, Navier–Stokes–\( \alpha \) models, ...)

Ψ (≥ 0, units of \( L^2 \)) is the dispersion function; can depend only on the invariant (symm[grad \( \mathbf{u} \)])\(^2\).
RRG theory: Reduction to nonlinear evolution eqs. (NEEs)

- **Example 1:** unidirectional weakly-nonlinear equation for shear waves in a hyperelastic, incompressible solid with $\Psi \propto (\text{symm}[\nabla u])^3$,

$$v_t + \frac{1}{2}(v^3)_x + \frac{1}{3}[(v_x)^3]_{xx} = 0 \quad (v = \text{strain}).$$


- **Example 2:** unidirectional weakly-nonlinear equation for acoustic waves in an inviscid, non-thermally conducting compressible fluid with $\Psi \propto (\text{symm}[\nabla u])^2$:

$$u_t + \epsilon \beta uu_x + \frac{1}{6}a_1[(u_x)^3]_{xx} = 0 \quad (u = \text{velocity}).$$

(Jordan & Saccomandi, *Proc. R. Soc. A*, 2012)

- **Today:** a new way of looking at these NEEs as members of a generalized KdV-like hierarchy of Hamiltonian PDEs.
Historical detour: The Korteweg–de Vries (KdV) eq.

- The NEE (Korteweg & de Vries, *Phil. Mag.*, 1895)

\[ \eta_t + c_0 \eta_x + \alpha \eta \eta_x + \beta \eta_{xxx} = 0 \]

describes long surface waves over shallow water with

\[ c_0 = \sqrt{gh}, \quad \alpha = \frac{3c_0}{2h}, \quad \beta = \frac{1}{6}c_0 h^2. \]

- Can be transformed to canonical form:

\[ u_t + uu_x + u_{xxx} = 0. \quad (1) \]

- Eqs. of inviscid, irrotational, incompressible fluids are Hamiltonian.

- KdV retains this structure, (1) extremizes the action \( \int dx dt \mathcal{L} \) with

\[ \mathcal{L}(\varphi; x, t) = \frac{1}{2} \varphi_x \varphi_t + \frac{1}{6} (\varphi_x)^3 - \frac{1}{2} (\varphi_{xx})^2, \quad \varphi_x = u. \]

\textbf{\(K^\#(n, m):\) A nonlinearly dispersive KdV hierarchy}

- Consider

\[ \mathcal{L}(\phi; x, t) = \frac{1}{2} \phi \phi_t + \frac{1}{(n+2)(n+1)} (\phi_x)^{n+2} - \frac{1}{m+1} (\phi_{xx})^{m+1}, \quad n, m > 0. \]

[KdV is \(n = m = 1\)]

- Then, action functional \(\int dxdt \mathcal{L}\) is extremized by

\[ \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \phi_t} \right) - \frac{\partial}{\partial x} \left( \frac{\partial \mathcal{L}}{\partial \phi_x} \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\partial \mathcal{L}}{\partial \phi_{xx}} \right) = 0. \]

- I.e., Euler–Lagrange eq. is

\[ \phi_{xt} + (\phi_x)^n \phi_{xx} + [(\phi_{xx})^m]_{xx} = 0. \]

- Let \(u(x, t) \equiv \phi_x(x, t)\), then

\[ u_t + u^n u_x + [(u_x)^m]_{xx} = 0. \]
Conserved quantities \([d(\cdot)/dt = 0]\)

- Legendre transformation: \(H = \frac{\partial \mathcal{L}}{\partial \varphi_t} \varphi_t - \mathcal{L}\), then total wave energy:

\[
H \equiv \int_{-\infty}^{+\infty} dx \, \mathcal{H} = \int_{-\infty}^{+\infty} dx \left[ -\frac{1}{(n+2)(n+1)} u^{n+2} + \frac{1}{m+1} (u_x)^{m+1} \right].
\]

- Total wave mass:

\[
M \equiv \int_{-\infty}^{+\infty} dx \frac{\partial \mathcal{L}}{\partial \varphi_t} = \frac{1}{2} \int_{-\infty}^{+\infty} dx \varphi_x = \frac{1}{2} \int_{-\infty}^{+\infty} dx \, u.
\]

- Total wave momentum:

\[
P \equiv -\int_{-\infty}^{+\infty} dx \frac{\partial \mathcal{L}}{\partial \varphi_t} \varphi_x = -\frac{1}{2} \int_{-\infty}^{+\infty} dx (\varphi_x)^2 = -\frac{1}{2} \int_{-\infty}^{+\infty} dx \, u^2.
\]

- But, no other conserved quantities of the form \(\int u^k\).
Connection to other nonlinearly dispersive KdV-like eqs.

- $K(n, m)$ eqs. [not Hamiltonian]:

$$u_t + (u^n)_x + (u^m)_{xxx} = 0.$$  


- $K^*(l, p)$ eqs. [Hamiltonian, $\mathcal{L}_{K^*} = \frac{1}{2} \varphi_x \varphi_t + \frac{(\varphi_x)^l}{l(l-1)} - \frac{1}{2} (\varphi_x)^p (\varphi_{xx})^2$]:

$$u_t + u^{l-2} u_x + u^p u_{xxx} + 2pu^{p-1} u_x u_{xx} + \frac{1}{2} p(p - 1)u^{p-2}(u_x)^3 = 0.$$  

(Cooper et al., Phys. Rev. E, 1993)

- $K^#(n, m)$ eqs. [Hamiltonian, $\mathcal{L}_{K^#} = \frac{1}{2} \varphi_x \varphi_t + \frac{(\varphi_x)^{n+2}}{(n+2)(n+1)} - \frac{(\varphi_{xx})^{m+1}}{m+1}$]:

$$u_t + u^n u_x + [(u_x)^m]_{xx} = 0.$$
Nonlinear dispersion leads to compactification

- $K(n, m)$ hierarchy $u_t + (u^n)_x + (u^m)_{xxx} = 0$ was introduced to study effects of nonlinear dispersion.

- As a result of the dispersive term’s degeneracy at $u = 0$, compact traveling wave solutions can be constructed, e.g., for $K(2, 2)$:

$$u(x, t) = \begin{cases} \frac{4c}{3} \cos^2 \left[ \frac{1}{4}(x - ct) \right], & |x - ct| \leq 2\pi \\ 0, & \text{else.} \end{cases}$$

- Why this is physically enticing: Interactions of solitons (e.g., KdV) are determined by the tail asymptotics...but what if there are no tails?
Reduction of $K^\#(n, m)$ to an ODE

- Let $u(x, t) = U(\xi)$, where $\xi = x - ct$ for some wave speed $c > 0$,
  
  $$-cU' + U^n U' + [(U')^m]'' = 0.$$  

- Two integrals can be performed:
  
  $$(U')^{m+1} = C_2 + C_1 U + \frac{(m+1)}{2m}cU^2 - \frac{(m+1)}{(n+1)(n+2)m}U^{n+2}.$$  

- Finally, implicit solution is
  
  $$\int \frac{dU}{[C_2 + C_1 U + U^2 (\kappa - \gamma U^n)]^{1/(m+1)}} = \pm \xi + \xi_0,$$

  $\kappa \equiv \frac{(m+1)}{2m}c, \quad \gamma \equiv \frac{(m+1)}{(n+1)(n+2)m}.$

- Only interested in localized waveforms such that $u, u' \to 0$ as $|x| \to \pm \infty \Rightarrow C_1 = C_2 = 0.$
Constructing a peakompacton solution

So,

\[ \int \frac{dU}{[U^2 (\kappa - \gamma U^n)]^{1/(m+1)}} = \pm \xi + \xi_0, \]  

which evaluates to a hypergeometric function (still implicit):

\[ \left( \frac{m+1}{m-1} \right) U(\kappa U^2)^{-1/(m+1)} \, _2F_1 \left[ \frac{1}{m+1}, \frac{m-1}{(m+1)n}, 1 + \frac{m-1}{(m+1)n}, \frac{\gamma}{\kappa} U^n \right] = \pm \xi + \xi_0. \]  

Notice that denominator in (3) vanishes at

\[ U = \left\{ 0, \left[ \frac{1}{2} (n + 1)(n + 2)c \right]^{\frac{1}{n}} \right\} \]  

but \( \int \) is non-divergent \( \Rightarrow \pm \xi + \xi_0 < \infty \) \( \Rightarrow \) piece together positive sol’n with \( U = 0 \).

Traveling wave solution is \( U = 0 \) for \( \xi \in (-\infty, -\xi_0] \cup [+\xi_0, +\infty) \), and \( U \) found by inverting (4) for \( \xi \in (-\xi_0, +\xi_0) \).

Can verify that \( U \in C^1(\mathbb{R}) \) and \( |U''| \to \infty \) as \( \xi \to 0 \: \) compact and peaked.
Peakompactons illustrated

Figure: $c = 0.75$ and (a) $n = 1, 2, 3$ (solid, dashed, dotted) and $m = 3$, (b) $n = 1$ and $m = 3, 4, 5$ (solid, dashed, dotted).
Leap-Frog scheme with filtering (w/ T. Kress, summer student from UNC)

- Semi-discretize with central differences in space (+ periodic BC):

\[
\frac{du_j}{dt} = - \left\{ \frac{2}{3} D_0 \left[ \frac{1}{n+1} u_j^{n+1} \right] + \frac{1}{3} u_j^n D_0 [u_j] \right\} + D^2 [(D_0 [u_j])^m] \equiv L_h [u_j].
\]

- Advance in time by “Leap-Frog 2–3” (Hyman, Proc. IMACS, 1979):

  Predictor: \( u_j^* = u_j^n + 2 \Delta t L_h [u_j^{n+1}] \),

  Corrector: \( u_j^{n+2} = \frac{1}{5} \left\{ u_j^n + 4 u_j^{n+1} + 4 \Delta t L_h [u_j^{n+1}] + 2 \Delta t L_h [u_j^*] \right\} \).

- Stabilize w/ filtered artificial viscosity (Cooper et. al, Phys. Rev. E, 2000):

\[
L_h [u_j] \mapsto L_h [u_j] + \text{ramp}_\text{spectral}_\text{filter} [2.5(\Delta x)D^2[u_j]],
\]

where \( \text{ramp}_\text{spectral}_\text{filter} [\cdot] \) eliminates the \( \frac{1}{3} \) lowest Fourier modes, keeps highest \( \frac{1}{3} \), ramps middle \( \frac{1}{3} \).
Overtaking collision in $K^\#(1, 3) \{u_t + uu_x + [(u_x)^3]_{xx} = 0\}$

- Decreasing $\Delta x$ [with $\Delta t = \frac{1}{2}(\Delta x)^3$ by stability condition] shows persistent features:
  - initial peakompactons survive collision,
  - 3rd one is “born”,
  - dispersive tail appears to be mesh-dependent.
Summary

- Proposed $K^\#(n, m)$ hierarchy of nonlinearly dispersive KdV-like PDEs with Hamiltonian structure.
- Constructed *peaked, compact* traveling wave solutions.
- Preliminary numerics suggest non-trivial, non-integrable dynamics.
- **Next:** Miura transformation? Stability of peakompactons? Qualitative properties of solutions?

Thank you for your attention!