On a Hierarchy of Nonlinearly Dispersive Generalized KdV Equations

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Recent Developments in Continuum Mechanics and Partial Differential Equations

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Introduction: Continua with an inherent length scale

- Balance laws of (local) continuum mechanics:

$$\underbrace{\dot{\varrho} + \varrho \nabla \cdot \boldsymbol{u} = 0,}_{\text{conservation of mass}} \qquad \underbrace{\varrho \dot{\boldsymbol{u}} = \nabla \cdot \mathbf{T} + \boldsymbol{b}.}_{\text{conservation of linear momentum}}$$

- "RRG" theory: let $\mathbf{T} = \mathbf{T}^{(1)} + \mathbf{T}^{(2)}$, where

$$\mathbf{T}^{(1)} = -\wp \mathbf{I} + 2\mu \,\mathrm{symm}[\nabla \boldsymbol{u}] + (\mu_{\mathrm{B}} - \tfrac{2}{3}\mu)(\nabla \cdot \boldsymbol{u})\mathbf{I},$$

$$\mathbf{T}^{(2)} = \varrho \Psi \left\{ 2\,\mathrm{symm}[\nabla \dot{\boldsymbol{u}}] + 2(\nabla \boldsymbol{u})^{\top} \nabla \boldsymbol{u} - 4(\mathrm{symm}[\nabla \boldsymbol{u}])^2 \right\}.$$

(Rubin et. al, *J. Appl. Phys.*, 1995)

- Ψ (≥ 0, units of L^2) is the dispersion function; can depend only on the invariant $(\mathrm{symm}[\mathrm{grad}\, \boldsymbol{u}])^2$.
- (Connections to "second-grade fluid" of Coleman & Gurtin, Navier–Stokes-α models, ...)

RRG theory: Reduction to nonlinear evolution eqs. (NEEs)

- Example 1: unidirectional weakly-nonlinear equation for shear waves in a hyperelastic, incompressible solid with $\Psi \propto (\text{symm}[\nabla \boldsymbol{u}])^3$,

$$v_t + \tfrac{1}{2}(v^3)_x + \tfrac{1}{3}[(v_x)^3]_{xx} = 0 \qquad (v = \text{strain}).$$

(Destrade & Saccomandi, *Phys. Rev. E*, 2006)

- Example 2: unidirectional weakly-nonlinear equation for acoustic waves in an inviscid, non-thermally conducting compressible fluid with $\Psi \propto (\text{symm}[\nabla \boldsymbol{u}])^2$:

$$u_t + \epsilon\beta u u_x + \tfrac{1}{6}a_1[(u_x)^3]_{xx} = 0 \qquad (u = \text{velocity}).$$

(Jordan & Saccomandi, *Proc. R. Soc. A*, 2012)

- **Today:** a new way of looking at these NEEs as members of a generalized KdV-like hierarchy of Hamiltonian PDEs.

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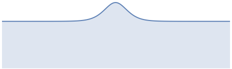
Historical detour: The Korteweg–de Vries (KdV) eq.

- The NEE (Korteweg & de Vries, *Phil. Mag.*, 1895)

$$\eta_t + c_0\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = 0$$

describes long surface waves over shallow water with

$$c_0 = \sqrt{gh}, \quad \alpha = \frac{3c_0}{2h}, \quad \beta = \frac{1}{6}c_0 h^2.$$



- Can be transformed to canonical form:

$$u_t + uu_x + u_{xxx} = 0. \qquad (1)$$

- Eqs. of inviscid, irrotational, incompressible fluids are Hamiltonian.
- KdV retains this structure, (1) extremizes the action $\int \mathrm{d}x\mathrm{d}t\,\mathcal{L}$ with

$$\mathcal{L}(\varphi; x, t) = \tfrac{1}{2}\varphi_x\varphi_t + \tfrac{1}{6}(\varphi_x)^3 - \tfrac{1}{2}(\varphi_{xx})^2, \quad \varphi_x = u.$$

(Gardner, *J. Math. Phys.*, 1971; Morrison, *Rev. Mod. Phys.*, 1998)

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$K^{\#}(n,m)$: A nonlinearly dispersive KdV hierarchy

- Consider

$$\mathcal{L}(\varphi;x,t) = \tfrac{1}{2}\varphi_x\varphi_t + \frac{1}{(n+2)(n+1)}(\varphi_x)^{n+2} - \frac{1}{m+1}(\varphi_{xx})^{m+1}, \quad n,m>0.$$

  [KdV is $n=m=1$]

- Then, action functional $\int \mathrm{d}x\mathrm{d}t\,\mathcal{L}$ is extremized by

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\varphi_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\varphi_x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial\mathcal{L}}{\partial\varphi_{xx}}\right) = 0.$$

- I.e., Euler–Lagrange eq. is

$$\varphi_{xt} + (\varphi_x)^n\varphi_{xx} + [(\varphi_{xx})^m]_{xx} = 0.$$

- Let $u(x,t) \equiv \varphi_x(x,t)$, then

$$u_t + u^n u_x + [(u_x)^m]_{xx} = 0.$$

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Conserved quantities $[\mathrm{d}(\cdot)/\mathrm{d}t = 0]$

- Legendre transformation: $\mathcal{H} = \frac{\partial\mathcal{L}}{\partial\varphi_t}\varphi_t - \mathcal{L}$, then total wave energy:

$$H \equiv \int_{-\infty}^{+\infty} \mathrm{d}x\,\mathcal{H} = \int_{-\infty}^{+\infty} \mathrm{d}x\,\left[-\tfrac{1}{(n+2)(n+1)}u^{n+2} + \tfrac{1}{m+1}(u_x)^{m+1}\right].$$

- Total wave mass:

$$M \equiv \int_{-\infty}^{+\infty} \mathrm{d}x\,\frac{\partial\mathcal{L}}{\partial\varphi_t} = \tfrac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}x\,\varphi_x = \tfrac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}x\,u.$$

- Total wave momentum:

$$P \equiv -\int_{-\infty}^{+\infty} \mathrm{d}x\,\frac{\partial\mathcal{L}}{\partial\varphi_t}\varphi_x = -\tfrac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}x\,(\varphi_x)^2 = -\tfrac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}x\,u^2.$$

- But, no other conserved quantities of the form $\int u^k$.

Connection to other nonlinearly dispersive KdV-like eqs.

- $K(n, m)$ eqs. [not Hamiltonian]:

$$u_t + (u^n)_x + (u^m)_{xxx} = 0.$$

(Rosenau & Hyman, *Phys. Rev. Lett.*, 1993)

- $K^*(l, p)$ eqs. [Hamiltonian, $\mathcal{L}_{K^*} = \frac{1}{2}\varphi_x\varphi_t + \frac{(\varphi_x)^l}{l(l-1)} - \frac{1}{2}(\varphi_x)^p(\varphi_{xx})^2$]:

$$u_t + u^{l-2}u_x + u^p u_{xxx} + 2pu^{p-1}u_x u_{xx} + \frac{1}{2}p(p-1)u^{p-2}(u_x)^3 = 0.$$

(Cooper *et al.*, *Phys. Rev. E*, 1993)

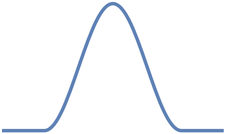
- $K^\#(n, m)$ eqs. [Hamiltonian, $\mathcal{L}_{K\#} = \frac{1}{2}\varphi_x\varphi_t + \frac{(\varphi_x)^{n+2}}{(n+2)(n+1)} - \frac{(\varphi_{xx})^{m+1}}{m+1}$]:

$$u_t + u^n u_x + [(u_x)^m]_{xx} = 0.$$

Nonlinear dispersion leads to compactification

- $K(n,m)$ hierarchy $u_t + (u^n)_x + (u^m)_{xxx} = 0$ was introduced to study effects of nonlinear dispersion.
- As a result of the dispersive term's degeneracy at $u = 0$, compact traveling wave solutions can be constructed, e.g., for $K(2,2)$:

$$u(x,t) = \begin{cases} \frac{4c}{3}\cos^2\left[\frac{1}{4}(x-ct)\right], & |x-ct| \leq 2\pi \\ 0, & \text{else.} \end{cases}$$



- Why this is physically enticing: Interactions of solitons (e.g., KdV) are determined by the tail asymptotics...but what if there are no tails?

Reduction of $K^{\#}(n,m)$ to an ODE

- Let $u(x,t) = U(\xi)$, where $\xi = x - ct$ for some wave speed $c > 0$,

$$-cU' + U^n U' + \left[(U')^m\right]'' = 0.$$

- Two integrals can be performed:

$$(U')^{m+1} = C_2 + C_1 U + \tfrac{(m+1)}{2m} cU^2 - \tfrac{(m+1)}{(n+1)(n+2)m} U^{n+2}.$$

- Finally, implicit solution is

$$\int \frac{\mathrm{d}U}{\left[C_2 + C_1 U + U^2\left(\kappa - \gamma U^n\right)\right]^{1/(m+1)}} = \pm\xi + \xi_0,$$

$\kappa \equiv \frac{(m+1)}{2m}c$, $\gamma \equiv \frac{(m+1)}{(n+1)(n+2)m}$.

- Only interested in localized waveforms such that $u, u' \to 0$ as $|x| \to \pm\infty \Rightarrow C_1 = C_2 = 0$.

Constructing a peakompacton solution

- So,

$$\int \frac{\mathrm{d}U}{\left[U^2\left(\kappa - \gamma U^n\right)\right]^{1/(m+1)}} = \pm\xi + \xi_0, \tag{3}$$

which evaluates to a hypergeometric function (still implicit):

$$\left(\frac{m+1}{m-1}\right) U(\kappa U^2)^{-1/(m+1)} \, {}_2F_1\left[\frac{1}{m+1}, \frac{m-1}{(m+1)n}, 1 + \frac{m-1}{(m+1)n}, \frac{\gamma}{\kappa} U^n\right] = \pm\xi + \xi_0. \tag{4}$$

- Notice that denominator in (3) vanishes at $U = \left\{0, [\frac{1}{2}(n+1)(n+2)c]^{\frac{1}{n}}\right\}$ but $\int$ is non-divergent $\Rightarrow \pm\xi + \xi_0 < \infty$ $\Rightarrow$ piece together positive sol'n with $U = 0$.
- Traveling wave solution is $U = 0$ for $\xi \in (-\infty, -\xi_0] \cup [+\xi_0, +\infty)$, and U found by inverting (4) for $\xi \in (-\xi_0, +\xi_0)$.
- Can verify that $U \in C^1(\mathbb{R})$ and $|U''| \to \infty$ as $\xi \to 0$: compact *and* peaked.

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Peakompactons illustrated

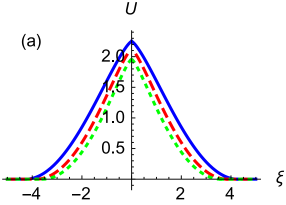
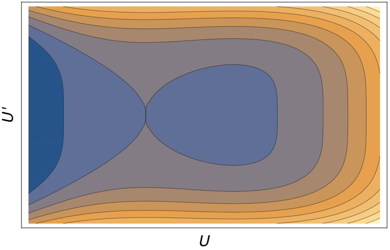


Figure: $c = 0.75$ and (a) $n = 1, 2, 3$ (solid, dashed, dotted) and $m = 3$, (b) $n = 1$ and $m = 3, 4, 5$ (solid, dashed, dotted).



Leap-Frog scheme with filtering (w/ T. Kress, summer student from UNC)

- Semi-discretize with central differences in space (+ periodic BC):

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = -\left\{ \tfrac{2}{3} D_0\left[\tfrac{1}{(n+1)} u_j^{n+1}\right] + \tfrac{1}{3} u_j^n D_0[u_j] \right\} + D^2\left[(D_0[u_j])^m\right] \equiv \mathcal{L}_h[u_j].$$

- Advance in time by "Leap-Frog 2–3" (Hyman, *Proc. IMACS*, 1979):

$$\text{Predictor}: \quad u_j^* = u_j^n + 2\Delta t \mathcal{L}_h[u_j^{n+1}],$$

$$\text{Corrector}: \quad u_j^{n+2} = \tfrac{1}{5}\left\{ u_j^n + 4u_j^{n+1} + 4\Delta t \mathcal{L}_h[u_j^{n+1}] + 2\Delta t \mathcal{L}_h[u_j^*] \right\}.$$

- Stabilize w/ filtered artificial viscosity (Cooper *et. al*, *Phys. Rev. E*, 2000):

$$\mathcal{L}_h[u_j] \mapsto \mathcal{L}_h[u_j] + \texttt{ramp\_spectral\_filter}\left[2.5(\Delta x) D^2[u_j]\right],$$

wher `ramp_spectral_filter[·]` eliminates the $\frac{1}{3}$ lowest Fourier modes, keeps highest $\frac{1}{3}$, ramps middle $\frac{1}{3}$.

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Overtaking collision in $K^{\#}(1,3)$ $\{u_t + uu_x + [(u_x)^3]_{xx} = 0\}$

movie

- Decreasing Δx [with $\Delta t = \frac{1}{2}(\Delta x)^3$ by stability condition] shows persistent features:
  - initial peakompactons survive collision,
  - 3rd one is "born",
  - dispersive tail appears to be mesh-dependent.

Summary

- Proposed $K^{\#}(n, m)$ hierarchy of nonlinearly dispersive KdV-like PDEs with Hamiltonian structure.
- Constructed *peaked*, *compact* traveling wave solutions.
- Preliminary numerics suggest non-trivial, non-integrable dynamics.
- **Next:** Miura transformation? Stability of peakompactons? Qualitative properties of solutions?
- **Ref.:** I.C. Christov, "On a hierarchy of nonlinearly dispersive generalized KdV equations," to appear in *Proc. Estonian Acad. Sci.*; arXiv:1501.01044.

Thank you for your attention!

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