Low Reynolds Number Fluid-Structure Interactions: Modeling the Response of Soft Microfluidic Devices

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Lab on a chip

- "The ability to perform laboratory operations on small scales using miniaturized (lab-on-a-chip) devices has many benefits. Designing and fabricating such systems is extremely challenging, but physicists and engineers are beginning to construct highly integrated and compact labs on chips with exciting functionality ..." – *Nature* **442** (2006)

- But there's much to learn ...

BUSINESS | TECH

Hot Startup Theranos Has Struggled With Its Blood-Test Technology

Silicon Valley lab, led by Elizabeth Holmes, is valued at $9 billion but isn't using its technology for all the tests it offers

– *Wall Street Journal* (Oct. 26, 2015)

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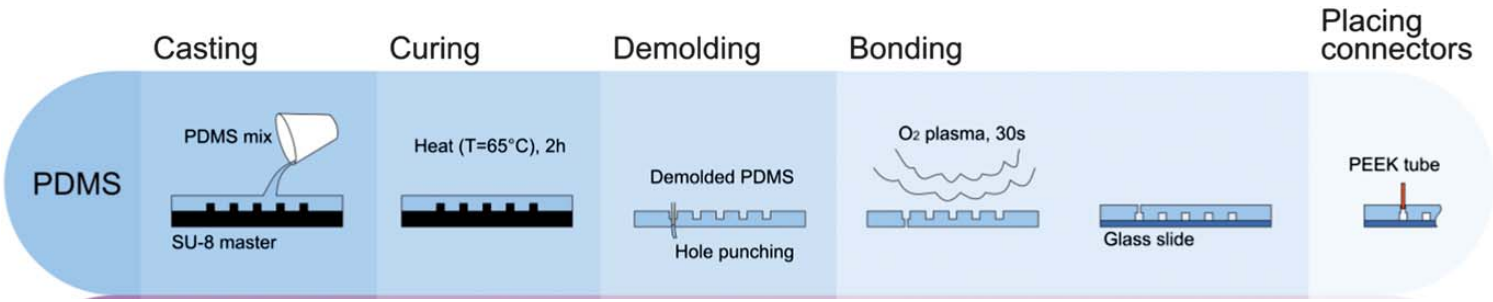
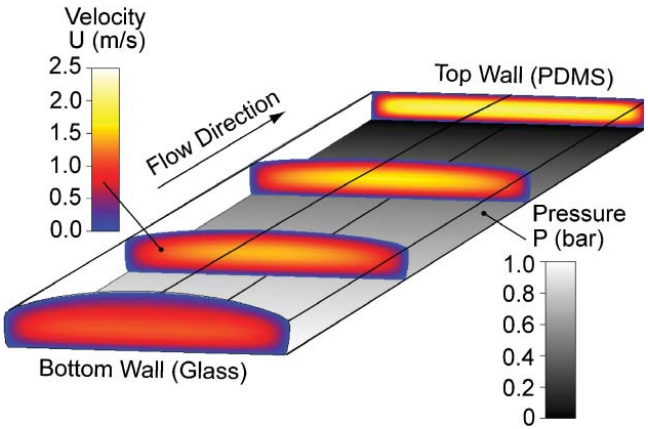
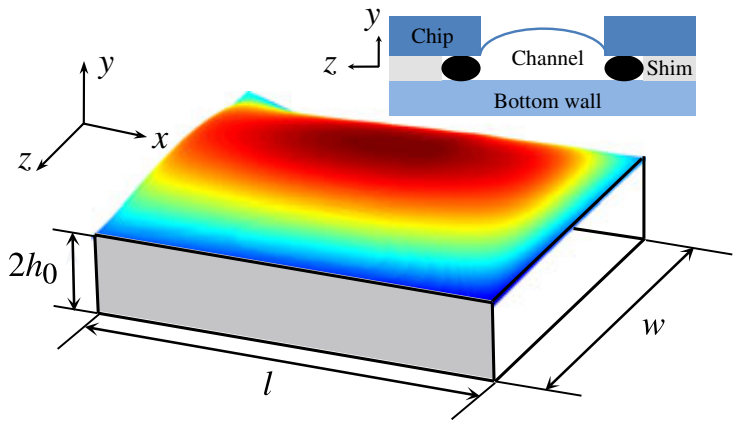
Fabrication and response of microfluidic devices

- Typical lab-on-a-chip device fabrication via replica molding of PDMS.
  (Sollier, *et al.*, *Lab Chip*, 2011)



- Resulting microchannels are soft. (Gervais, *et al.*, *Lab Chip*, 2006; Ozsun, *et al.*, *JFM*, 2013)





- For a rigid conduit, $q = R_h^{-1}\Delta p$ ($\sim$ Poiseuille).
- What if flow and deformation are coupled? $q = f(\Delta p)$?

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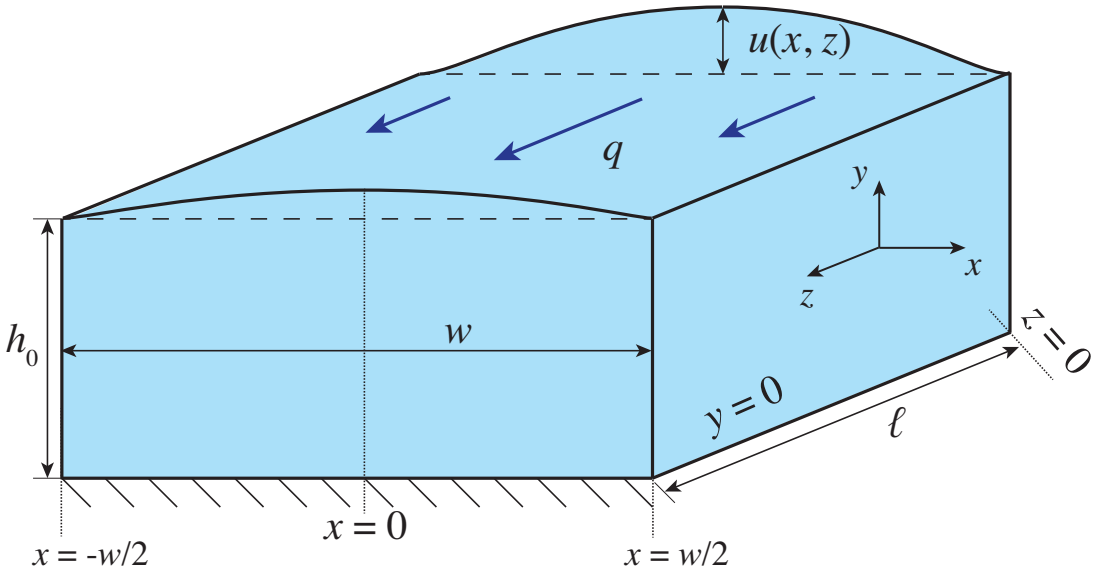
Problem formulation: Static response



- For $(h_0/\ell)Re \ll 1$, the Navier–Stokes equations for the fluid become:

$$\text{Stokes} \begin{cases} 0 = -\nabla p + \mu \nabla^2 \boldsymbol{v}, \\ \nabla \cdot \boldsymbol{v} = 0, \end{cases}$$

$$\text{BCs} \begin{cases} \boldsymbol{v} = \boldsymbol{0} \text{ at } y = 0,\ h(x,z). \\ h(x,z) = h_0 + u(x,z). \end{cases}$$

- For the top wall: Kirchoff–Love equation for bending of thin plates,

$$\underbrace{\left[\frac{Et^3}{12(1-\nu^2)}\right]}_{\equiv B} \nabla_{\parallel}^2 \nabla_{\parallel}^2 u = p, \qquad u = \frac{\partial u}{\partial x} = 0 \text{ at } x = \mp\frac{w}{2}; z = 0, \ell.$$

- All other walls are rigid.

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Asymptotics

- Perturbation expansion, $\{\boldsymbol{V}, P, U\} \sim \sum_i \sum_j \epsilon^i \delta^j \{\boldsymbol{V}, P, U\}_{ij}$, assuming $0 < \underbrace{h_0/\ell}_{\equiv\epsilon} \ll \underbrace{h_0/w}_{\equiv\delta} \ll 1$. [*Capital letters = dimensionless.*]
- At the leading order, we get the usual lubrication approximation:

$$V_{Z,00}(X, Y, Z) = -\frac{1}{2}\frac{\mathrm{d}P_{00}}{\mathrm{d}Z}[H(X, Z) - Y]Y.$$

- Then, the flow rate follows

$$Q = -\frac{1}{12}\frac{\mathrm{d}P_{00}}{\mathrm{d}Z}\int_{-1/2}^{+1/2} H(X, Z)^3 \,\mathrm{d}X.$$

- Still need to know $H(X, Z)$ to finish the calculation ... at leading order, plate eq. reduces to beam eq. in the streamwise cross-section ($H = 1 + \beta U$):

$$\frac{\partial^4 U_{00}}{\partial X^4} = P_{00}(Z), \qquad \beta := \frac{u_c}{h_0} = \frac{h_0^3 \Delta p}{B\delta^4} = \mathcal{O}(1).$$

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Flow rate–pressure drop relation

- Putting all the pieces together:

$$Q = \frac{P(Z)}{12(1-Z)}\left[1 + \frac{\tilde{\beta}}{20}P(Z) \cancel{\frac{\tilde{\beta}^2}{630}P(Z)^2} + \cancel{\frac{\tilde{\beta}^3}{48,048}P(Z)^3}\right]$$

  where $\tilde{\beta} = \beta/24$ and $\Delta P = P(0)$.

- Deformation ($\beta \uparrow$) increases Q for a fixed ΔP, decreases P overall:

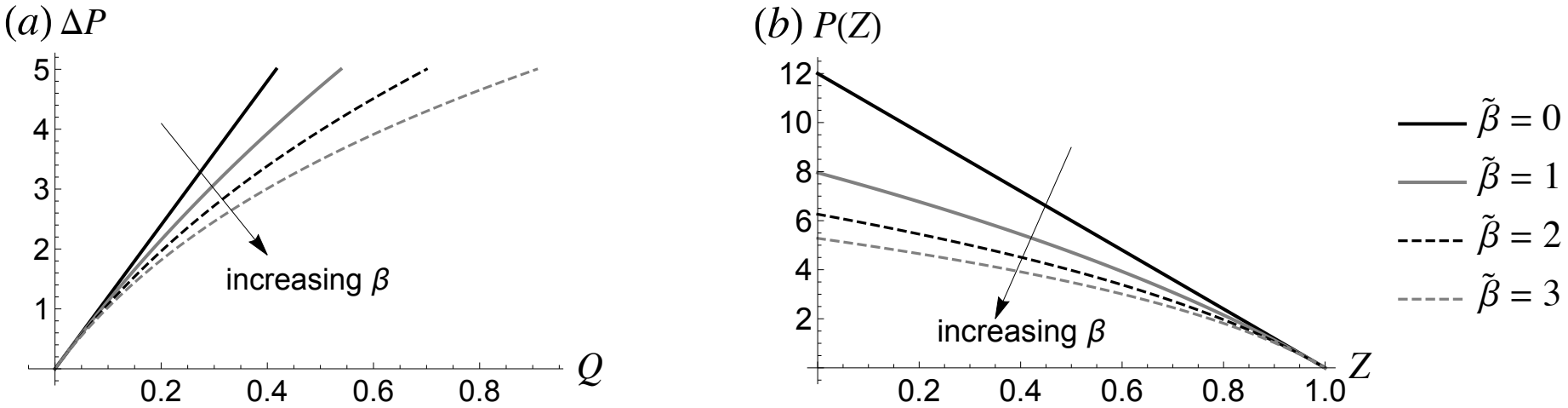

Figure: (*a*) ΔP–Q, different $\tilde{\beta}$; (*b*) $P(Z)$, $Q = 1$, different $\tilde{\beta}$.

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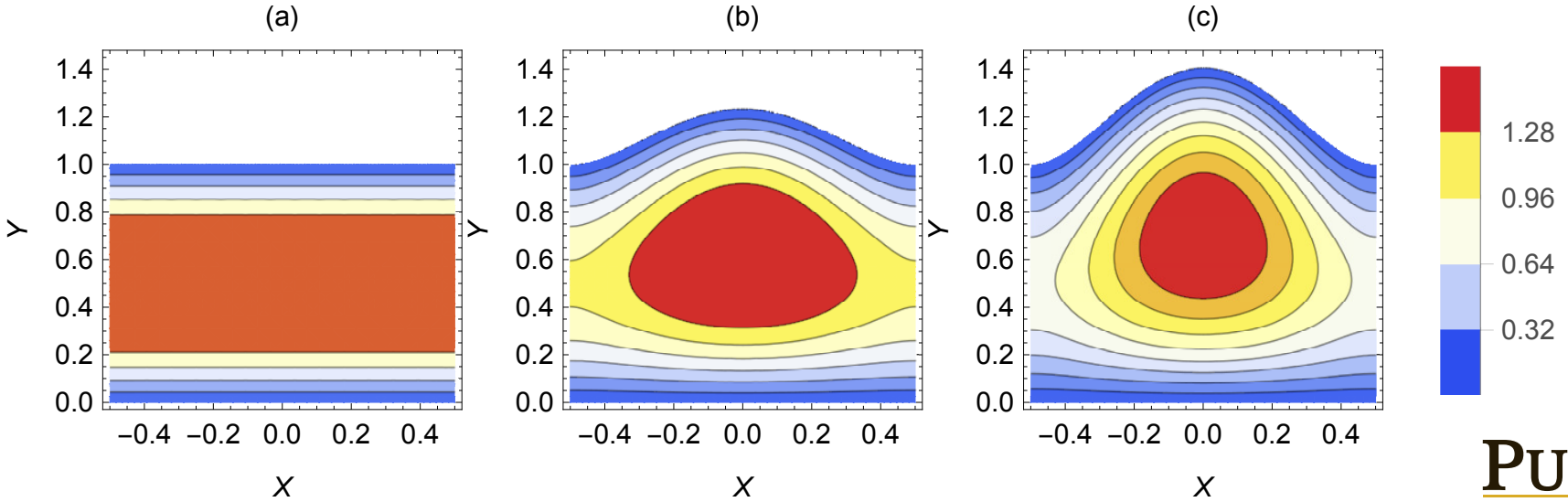
Explicit expression for the velocity field

- Even for $\beta = \mathcal{O}(1)$, keeping only terms up to P^2 is a good approx.,

$$P(Z) = -\frac{10}{\tilde{\beta}}\left[1 - \sqrt{1 + \frac{12}{5}\tilde{\beta}Q(1-Z)}\right].$$

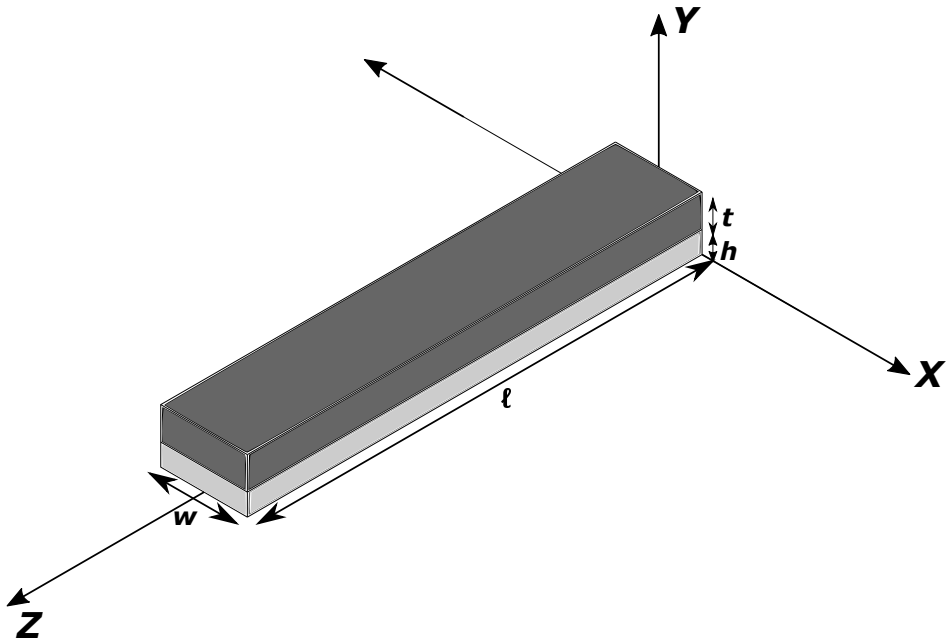
- Hence,

$$V_Z(X,Y,Z) = 6Q\left[1 + \frac{12}{5}\tilde{\beta}Q(1-Z)\right]^{-1/2} \times \left\{1 + \tilde{\beta}P(Z)\left(X+\tfrac{1}{2}\right)^2\left(X-\tfrac{1}{2}\right)^2 - Y\right\} Y.$$

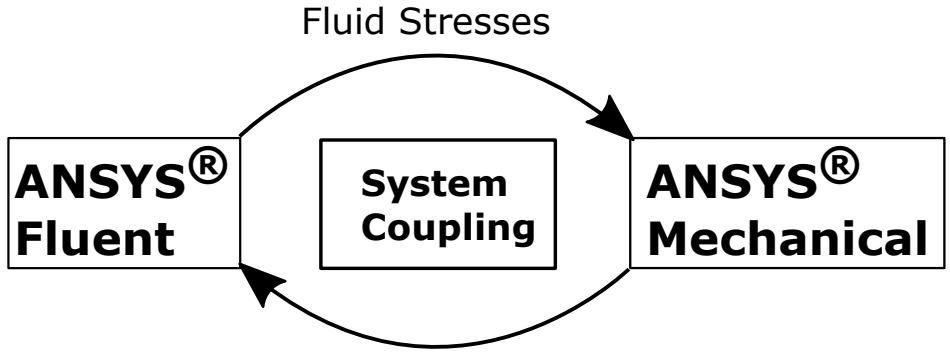


Two-way fluid–structure interaction simulations

- Set up simulations to benchmark against (Ozsun, *et al.*, *JFM*, 2013):
  - Case 1: $\ell \times w = 15.5\,\text{mm} \times 1.7\,\text{mm}$, $h_0 = 244\,\mu\text{m}$, $t_0 = 200\,\mu\text{m}$ $(t_0/w \simeq 0.12)$; $E = (1.6 \pm 0.2)\,\text{MPa}$, $\nu_s = 0.499$.
  - Case 2: $\ell \times w = 15.5\,\text{mm} \times 1.7\,\text{mm}$, $h_0 = 155\,\mu\text{m}$, $t_0 = 605\,\mu\text{m}$ $(t_0/w \simeq 0.35)$; $E = (1.6 \pm 0.2)\,\text{MPa}$, $\nu_s = 0.499$.



- FSI simulations using ANSYS/Fluent:
  - Partitioned approach w/ "system coupling" module.
  - BCs as discussed before; iterate to steady state.
  - Large-deformations are considered; grid-independence was verified.



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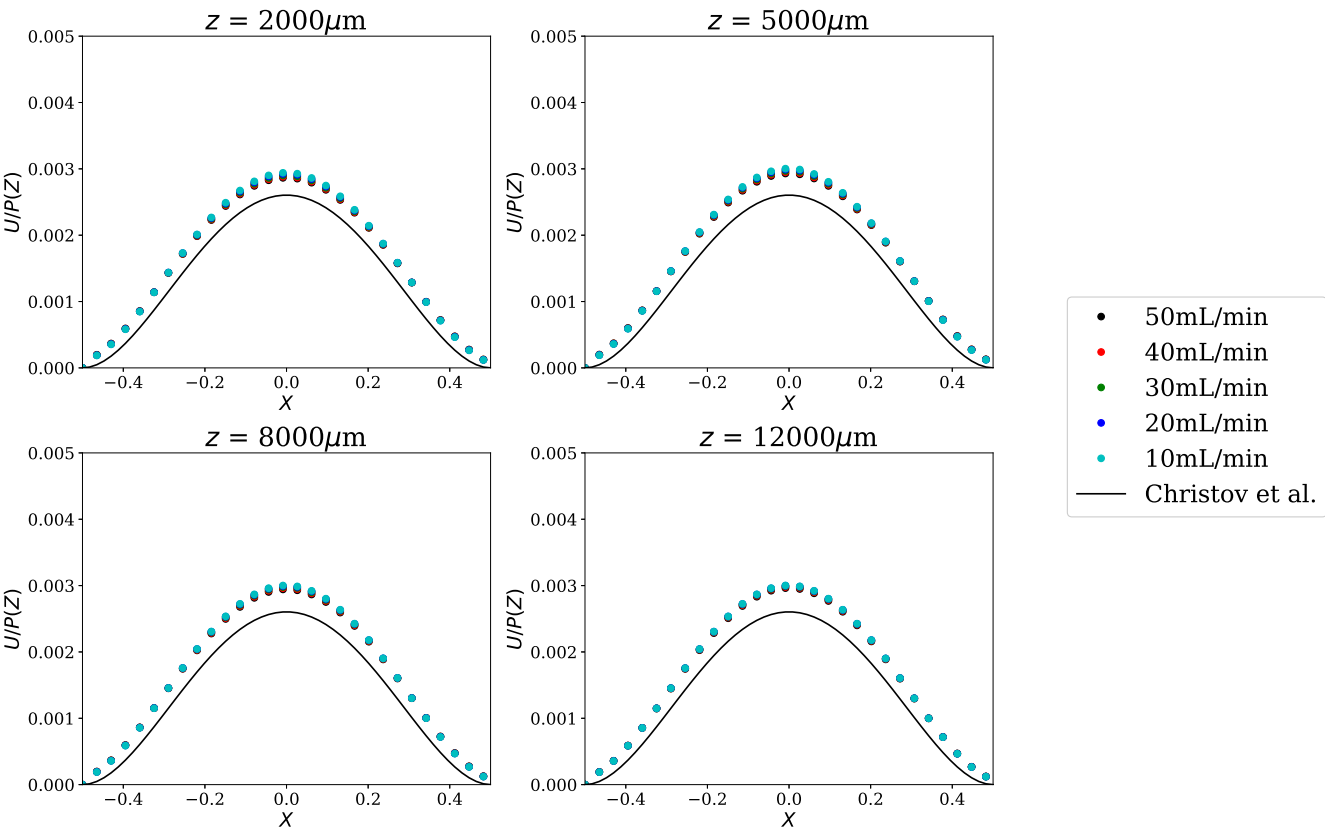
Top-wall deformation: FSI simulations vs. thin-plate theory

- The leading-order deformation shape shape is self-similar (in Z) :

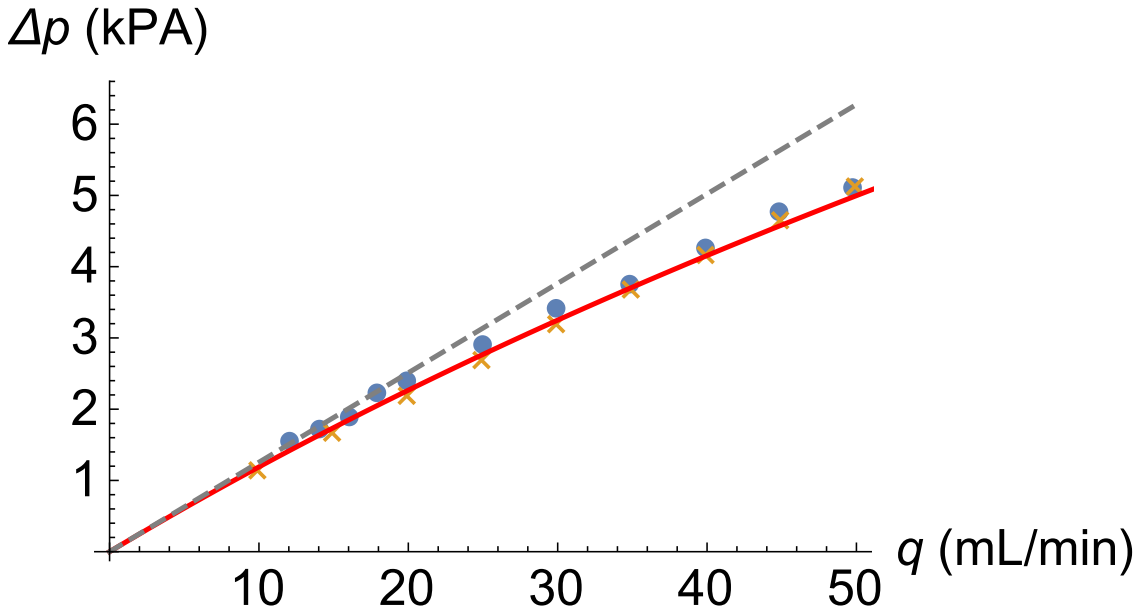
$$\frac{\partial^4 U}{\partial X^4} = P(Z) \quad \Rightarrow \quad \frac{U(X,Z)}{P(Z)} = \frac{1}{24}\left(X - \tfrac{1}{2}\right)^2\left(X + \tfrac{1}{2}\right)^2.$$

- FSI simulations of the "S4" case from (Ozsun, *et al.*, *JFM*, 2013) verify this:



Case 1: $t_0/w \simeq 0.12$ ($\approx$ a thin plate).

Flow rate–pressure drop: Experiment, numerics & theory



- •s: "S4" data (Ozsun, *et al.*, *JFM*, 2013); ×s: our FSI simulation.
- Dashed line – rectangular conduit (∼Poiseuille), $q = h_0^3 w \Delta p/(12\mu\ell)$.
- Solid curve – (new) perturbative theory:

$$q = \frac{h_0^3 w\, \Delta p}{12\mu\ell}\left[1 + \frac{1}{480}\frac{w^4}{Bh_0}\Delta p + \frac{1}{362,880}\left(\frac{w^4}{Bh_0}\right)^2(\Delta p)^2 + \frac{1}{664,215,552}\left(\frac{w^4}{Bh_0}\right)^3(\Delta p)^3\right].$$

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Mindlin's thick-plate-bending theory

- Governing equations (Mindlin, *ASME J. Appl. Mech.*, 1951) in terms of deformation u, rotation of the normal ϕ, stress resultants $\mathcal{Q}$ and $\mathcal{M}$.
- Equilibrium conditions (perturbative approach $\Rightarrow$ keep only $\partial/\partial x$):

$$\frac{\partial \mathcal{Q}}{\partial x} + p(z) = 0,$$
$$\frac{\partial \mathcal{M}}{\partial x} - \mathcal{Q} = 0.$$

- Constitutive relations are

$$\mathcal{M} = B\frac{\partial \phi}{\partial x},$$
$$\mathcal{Q} = \kappa G t \left(\frac{\partial u}{\partial x} + \phi\right),$$

where κ is a "shear correction factor," and $G = E/[2(1+\nu_s)]$ is elastic shear modulus.

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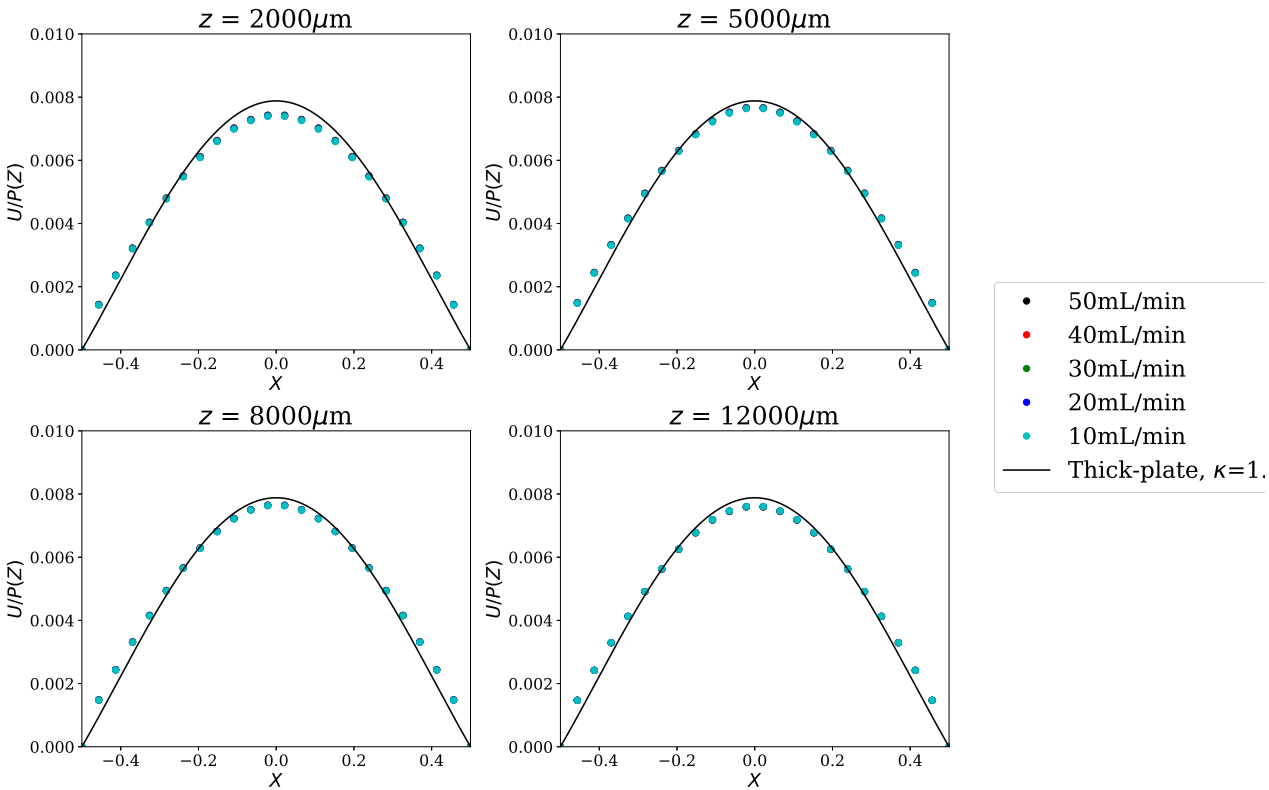
Top-wall deformation: Simulations vs. thick-plate theory

- Following (Mindlin, *ASME J. Appl. Mech.*, 1951) thick-plate theory, we get another leading-order self-similar (in Z) deformation shape:

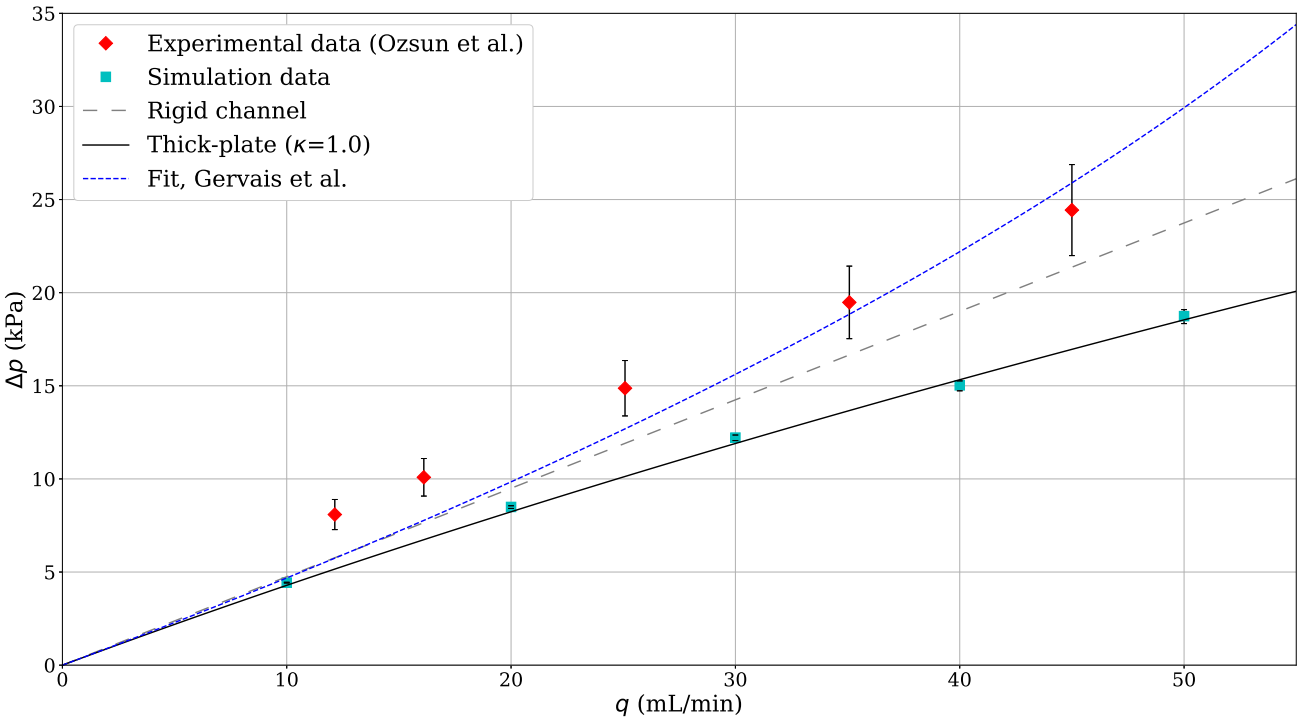
$$\frac{U(X,Z)}{P(Z)} = \frac{1}{24}\left(\frac{1}{4} - X^2\right)\left[\frac{2(t/w)^2}{\kappa(1-\nu_s)} + \left(\frac{1}{4} - X^2\right)\right].$$

- FSI simulations of the "S5" case from (Ozsun, *et al.*, *JFM*, 2013) verify this:



Case 2: $t_0/w \simeq 0.35$ ($\not\approx$ a thin plate).

Flow rate–pressure drop: Experiment, numerics & theory



- Case 2: "S5" data (Ozsun, *et al.*, *JFM*, 2013) vs. our simulations & theory.
- Solid curve – (new) perturbative theory for thick plates:

$$q = \frac{h_0^3 w\,\Delta p}{12\mu\ell}\left\{1 + \left[\frac{1}{480} + f_1(t/w)\right]\frac{w^4}{Bh_0}\Delta p + \left[\frac{1}{362,880} + f_2(t/w)\right]\left(\frac{w^4}{Bh_0}\right)^2(\Delta p)^2 + \left[\frac{1}{664,215,552} + f_3(t/w)\right]\left(\frac{w^4}{Bh_0}\right)^3(\Delta p)^3\right\}$$

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Summary

- Systematically derived flow rate–pressure drop relations for a deformable shallow microchannel, explicitly showing 3 dependencies:

$$q \approx \frac{h_0^3 w \Delta p}{12 \mu \ell} \left[1 + \mathfrak{F} \left(\frac{\Delta p}{E}, \frac{h_0}{w}, \frac{t}{w} \right) \right]$$

- Refs: I.C.C., V. Cognet, T.C. Shidhore, H.A. Stone, "Flow rate–pressure drop relation for deformable shallow microfluidic channels," in rev. for *J. Fluid Mech.*, arXiv:1712.02687

  T.C. Shidhore, I.C.C., "Static response of deformable microchannels: A comparative modelling study," in rev. for *J. Phys.: Condens. Mat.*, arXiv:1709.03002.

Thank you for your attention!

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