Anomalous Diffusion in Granular Flow: Fractional Kinetics or Intermediate Asymptotics?

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Granular Flow from a Dynamical Systems Perspective - Part II of II
2013 SIAM Conference on Dynamical Systems

May 20, 2013
The big picture

“Can we develop a general theory of the dynamics of turbulent flows and the motion of granular materials? So far, such ‘nonequilibrium systems’ defy the tool kit of statistical mechanics, and the failure leaves a gaping hole in physics.” — “So Much More to Know ...” (2005)

Motivation:

- Flowing granular materials — a complex system far from equilibrium.
  - No “general theory” but simple models can be enlightening.
- Diffusion, rheology, etc. in/of granular flow are not well understood.

Today:

1. How/why we observe “diffusion” in granular flows.
2. Anomalous scalings from “normal” diffusion?
Diffusion of a “granular pulse” in a tumbler

- A band of colored particles in a monodisperse granular mixture diffuses axially as the container is rotated.
  

- Many regimes of flow in a tumbler...restrict to continuous flow.

- Thin surface shear layer ⇒ velocity fluctuations ⇒ lateral diffusion.

(Khan et al., New J. Phys. 2011)
Anomalous macroscopic scalings

- **Experimental** concentration profiles of an axially diffusing pulse of dyed black salt grains in white salt grains: (Khan & Morris, *Phys. Rev. Lett.* 2005)

- **Experimental** concentration profiles of an axially diffusing pulse 4 mm spherical beads in bed of 2mm beads: (Finger et al., *Phys. Rev. E* 2009)
An explanation: Fractional kinetics

- For a random walk with characteristic waiting that diverges, but with finite jump length variance,

\[ 0\mathcal{D}_t^{2\alpha} c - \frac{t^{-2\alpha}}{\Gamma(1 - 2\alpha)} c(x, 0) = D_\alpha \frac{\partial^2 c}{\partial x^2}, \]

where

\[ 0\mathcal{D}_t^{2\alpha} c = \frac{1}{\Gamma(1 - 2\alpha)} \frac{\partial}{\partial t} \int_0^t dt' \frac{c(x, t')}{(t - t')^{2\alpha}}, \quad 0 < \alpha < 1/2. \]

- Self-similar solution

\[ c(x, t) = \frac{1}{\sqrt{4\pi D_\alpha t^{2\alpha}}} F \left( \frac{x}{\sqrt{4D_\alpha t^{2\alpha}}} \right), \quad F(\xi) = H^{2,0}_{1,2} \left[ \xi^2 \bigg| \frac{(1-\alpha,2\alpha)}{(0,1),\left(\frac{1}{2},1\right)} \right], \]

where \( H \) is Fox’s \( H \)-function (Fox, *Trans. Amer. Math. Soc.* 1961).

- \( \langle x^2(t) \rangle = \frac{2D_\alpha}{\Gamma(1+2\alpha)} t^{2\alpha}, \quad 0 < \alpha < 1/2 \quad \text{... subdiffusion} \quad \Rightarrow \text{must check in exp’t.} \)
The paradox: Normal microscopic scalings

- **Experimental** MSD, kurtosis from x-ray imaging (Khan et al., New J. Phys. 2011)

- **Simulation** of 5 mm beads axially diffusing into 10 mm beads:
Intermediate asymptotics of diffusion

- Consider dimensionless \([X = x/L, \ T = t/(L^2/D_0)]\) diffusion equation

\[
\frac{\partial C}{\partial T} = \frac{\partial}{\partial X} \left( D \frac{\partial C}{\partial X} \right), \quad \frac{\partial C}{\partial X} \bigg|_{X=\pm1} = 0, \quad C(X, 0) = C_i(X).
\]

- In general, \(D = D(C, X, \cdots)\).

- \(D = 1\): random walk with \(\langle (\Delta X)^2 \rangle \propto t\). (Einstein, *Ann. Phys.* 1905)
  - C-dependent jump probability leads to \(D(C)\). (Boon & Lutsko, *EPL* 2007)

- Similarity solution of the form \(C(X, T) = T^{-1/2} \mathcal{C}(XT^{-1/2})\).

Optimal self-similar solution (I)

- Let $\tau = T + T^*$ and $\eta = (X + X^*)/\sqrt{4(T + T^*)}$, then diffusion eq. becomes

$$\frac{\partial C}{\partial \tau} = \frac{\eta \partial C}{2 \partial \eta} + \frac{1}{4} \frac{\partial^2 C}{\partial \eta^2}.$$ 

- Separate variables as $C(\tau, \eta) = \tau^{-\lambda} g_{\lambda}(\eta)$ to get

$$g''_{\lambda} + 2\eta g'_{\lambda} + 4\lambda g_{\lambda} = 0 \quad \Rightarrow \quad g_{\lambda}(\eta) = e^{-\eta^2} \mathcal{H}_n(\eta), \quad \lambda = \frac{n + 1}{2}.$$ 

- For some $A_n$ determined from the I.C., the general solution is

$$C(\eta, T) = e^{-\eta^2} \sum_{n=0}^{\infty} \frac{A_n}{(T + T^*)^{\frac{n+1}{2}}} \mathcal{H}_n(\eta).$$ 


- But...I didn’t tell you what $T^*$ and $Z^*$ are!
Optimal self-similar solution (II)

- Pick $T^*$ and $X^*$ so that $A_1 = A_2 = 0$, then solutions for arbitrary initial data can be well-approximated (for $D = 1$ and $T > T^*$) by the point-source similarity solution

$$C(X, T) = \frac{1}{\sqrt{4\pi(T + T^*)}} \exp \left\{ -\frac{(X + X^*)^2}{4(T + T^*)} \right\} + O \left( \frac{1}{T^2} \right)$$

provided that

$$X^* = -\frac{M_1}{M_0}, \quad T^* = \frac{1}{2} \left[ \frac{M_2}{M_0} - \left( \frac{M_1}{M_0} \right)^2 \right],$$

and $M_k := \int_{-\infty}^{+\infty} C_i(X) X^k \, dX$.

- Note: $Z^* = 0$ in this talk (symmetric initial conditions only).
Time-dependent anomalous collapse exponents

- Data collapses under rescaling $C(X, T) \rightarrow C(X/T^\alpha, T) T^\alpha$.
- $\Rightarrow$ we are balancing $T^\alpha$ with $\sqrt{T + T^*}$, meaning

$$\alpha(T) \simeq \frac{\ln(T + T^*)}{2 \ln T} = \left\{ \begin{array}{ll}
\frac{1}{\ln T} \left[ \ln T^* + \frac{T}{2T^*} + O(T^2) \right], & T \rightarrow 0, \\
\frac{1}{2} + \frac{1}{\ln T} \left[ \frac{T^*}{2T} + O(T^{-2}) \right], & T \rightarrow \infty.
\end{array} \right.$$
Concentration-dependent diffusivity in bidisperse mixtures

- Suppose $C$ is the concentration of larger particles, then $D(C) = 1 + (C - 1/2)$. (Ristow & Nakagawa, Phys. Rev. E 1999)

- Seeking a self-similar solution $C(X, T) = T^{-\alpha} \zeta(\zeta)$ with $\zeta = T^{-\alpha} X$:

\[
0 = \zeta + \zeta' + \frac{1}{2\alpha} T^{1-2\alpha} \zeta'' + \frac{1}{2\alpha} T^{1-3\alpha} (\zeta^2)''.
\]

1. If $\alpha = 1/3$, then $T^{1-2\alpha} = T^{1/3} \ll 1$ for $T \ll 1$.

2. If $\alpha = 1/2$, then $T^{1-3\alpha} = T^{-1/2} \ll 1$ for $T \gg 1$.

- Q: Find $\alpha(T)$ in the crossover regime? [Christov, Eggers, Stone (in prep)]
Summary

○ What we have done:
  ▶ Everyone can be right! (Proposed a resolution to the granular diffusion “paradox.”)
  ▶ Analytical foundations for anomalous exponents in Fickian diffusion.

○ What we haven’t done:
  ▶ Find an asymptotic solution matching across both scaling regimes in the nonlinear case?
  ▶ Take into account the disparate flow regimes (thin shear layer on top of bulk in solid body rotation)?

Thank you for your attention!