Multiple-scale asymptotics of plane waves in media with variable phase speed

Ivan C. Christov

School of Mechanical Engineering
Purdue University

In collaboration with Prof. C.I. Christov,
Departments of Mathematics and Physics, University of Louisiana at Lafayette

Applied Mathematics Colloquium
Department of Mathematical Sciences
New Jersey Institute of Technology

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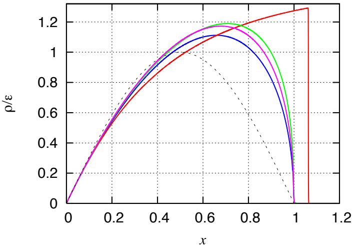
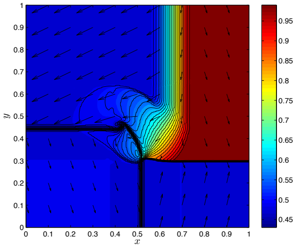
Cumulative effects in wave propagation (2006–present)

- Nonlinear acoustics (compressible, inviscid, irrotational fluids in 1D),

$$[1-\epsilon(\gamma-1)\phi_t]\phi_{xx} - 2\epsilon\phi_x\phi_{tx} - \phi_{tt} = \tfrac{1}{2}\epsilon^2(\gamma+1)(\phi_x)^2\phi_{xx}.$$

(I.C.C, Jordan & C.I.C., *Phys. Lett. A*, 2006; I.C.C., C.I.C. & Jordan, *Q. J. Math. Appl. Mech.*, 2007; I.C.C. & Jordan, *J. Acoust. Soc. Am.*, 2015)





- Hyperbolic systems of conservation laws (I.C.C. & Popov, *JCP*, 2008).
- **Today:** Plane waves under a variable phase speed
(I.C.C. & C.I.C., *J. Phys. A*, 2011; *submitted* 2016).

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Waves 101

What is a wave?

Most people assume that a wave, being central to all the phenomena we observe, has a uniform definition. But defining this basic concept isn't so easy.

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- To quote the SCOTUS,
  "we know it when we see it."
- Example:
  $u(x,t) = A\,\mathrm{Re}\,\{\exp\left[\mathrm{i}(kx - \omega t + \phi)\right]\}$,
  wavenumber k (m^{-1}),
  angular frequency ω (rad/s),
  phase speed $c = \omega/k$ (m/s),
  frequency $f = 2\pi/\omega$ (s^{-1}),
  period $T = 1/f = 2\pi/\omega$ (s),
  wavelength $\lambda = 2\pi/k$ (m),
  phase ϕ.

(Wikipedia, Public Domain)

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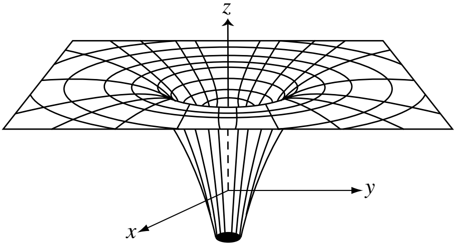
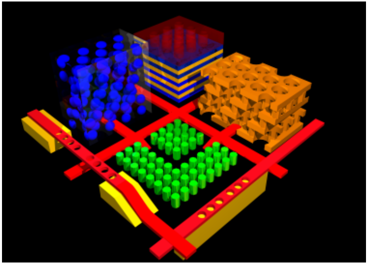
"Molding the Flow of Light"





(L) T.-P. Cheung, "Relativity, Gravitation, and Cosmology," Oxford Univ. Press, 2005

(R) J.D. Joannopoulos *et al.*, "Photonic Crystals," Princeton Univ. Press, 2008

- speed of light $c \to c(\vec{x})$ due to bending of space (GR), or
- $\mu_0 \to \mu(\vec{x})$ ($\varepsilon_0 = const.$ or vice versa) due manipulation of the path of light (photonics), $c \equiv 1/\sqrt{\varepsilon\mu}$.
  μ is the permeability of free space,
  ϵ is the permittivity of free space.

Maxwell's Eqs. with variable permissivity or permeability

- Maxwell's equations in the absence of sources are

$$\frac{\partial^2\{\vec{E},\vec{B}\}}{\partial t^2} = -\frac{1}{\{\varepsilon,\mu\}}\nabla\times\left(\frac{1}{\{\mu,\varepsilon\}}\nabla\times\{\vec{E},\vec{B}\}\right).$$

- An Advanced Calculus student's nightmare...

$$\frac{\partial^2\{\vec{E},\vec{B}\}}{\partial t^2} = \frac{1}{\{\varepsilon,\mu\}}\{\vec{E},\vec{B}\}\cdot(\nabla\nabla-\mathbb{I}\nabla^2)\frac{1}{\{\mu,\varepsilon\}}+\frac{1}{\{\varepsilon,\mu\}}\nabla\cdot\left(\frac{1}{\{\mu,\varepsilon\}}\nabla\{\vec{E},\vec{B}\}\right).$$

- Three cases can be unified as

$$\frac{\partial^2\vec{P}}{\partial t^2} = c^{2-\alpha}\vec{P}\cdot\left(\nabla\nabla-\mathbb{I}\nabla^2\right)c^{\alpha}+c^{2-\alpha}\nabla\cdot\left(c^{\alpha}\nabla\vec{P}\right),$$

where $\alpha = 2$ corresponds to $\mu_0 = \mu(\vec{x})$ ($\varepsilon_0 = const.$ or vice versa) and $\alpha = 1$ corresponds to $\mu_0 = \mu_0 f(\vec{x})$ *and* $\varepsilon_0 = \varepsilon_0 f(\vec{x})$.

Spherically-symmetric transverse waves

- $P_r = 0$ (i.e., $E_r = B_r = 0$), no zenith angle ϕ dependence.
- We make the transformation $P_\theta(r,\theta,t) = Q_\theta(r,t)/(r\sin\theta)$,

$$\frac{\partial^2 Q_\theta}{\partial t^2} - c^{2-\alpha}(r)\frac{\partial}{\partial r}\left[c^\alpha(r)\frac{\partial Q_\theta}{\partial r}\right] + \frac{\alpha}{r}c(r)c'(r)Q_\theta = 0.$$

- Finally, dropping θ subscript,

$$\frac{\partial^2 Q}{\partial t^2} = c^2(r)\frac{\partial^2 Q}{\partial r^2} + \alpha c'(r)c(r)\frac{\partial Q}{\partial r} - \frac{\alpha}{r}c(r)c'(r)Q, \quad \alpha \in \{1,2\}.$$

- **So, how do we understand cummulative effects on plane waves in media with variable c?** (Well, depends on what c is but...)
  - The method of multiple scales!

    (Cole, *Perturbation Methods in Applied Mathematics*, 1968; Kevorkian & Cole, *Multiple Scale and Singular Perturbation Methods*, 1996)

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Gravitational effects on the speed of light

- According a model due to Einstein (*Ann. Phys.*, 1911; *Science*, 1936),

$$c = c(r) = c_0\left(1 + \frac{\Phi(r)}{c_0^2}\right), \qquad \Phi(r) = -\frac{GM}{r},$$

  where $G = 6.67300 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$, $c_0 = 299,792,458$ m/s.

- A correction arises due to effect of the spatially-varying fundamental tensor in GR:

$$c(r) = c_0\left(1 + 2\frac{\Phi(r)}{c_0^2}\right) = c_0\left(1 - \frac{\delta}{r/r_0}\right), \qquad \delta := \frac{2GM/c_0^2}{r_0},$$

  where δ is the ratio of the *Schwarzschild radius* to the actual radius of the emitting body ("deviation from a flat spacetime").

- $\delta \geq 1 \Rightarrow$ black hole, so $\delta \ll 1$ for most objects.

Setting up the fast/slow scales

- Dimensionless temporal and spatial variables

$$\tau = t/(\omega_0^{-1}), \quad x = (r - r_0)/(c_0/\omega_0) = \epsilon^{-1}(r/r_0 - 1) \;\Leftrightarrow\; r = r_0(1+\epsilon x),$$

where $\epsilon := (c_0/\omega_0)/r_0 \ll 1$, i.e., wavelength is small compared to emitting body radius.

- Introducing a "long" spatial variable $x_1 = \epsilon x$,

$$c(r) = c_0 \hat{c}(x_1), \qquad \hat{c}(x_1) := 1 - \frac{\delta}{1 + x_1}.$$

- Partial derivatives transform as $\partial_t = \omega_0 \partial_\tau$, $\partial_r = (r_0\epsilon)^{-1}\partial_x$, and

$$\frac{\partial^2 q}{\partial \tau^2} = \hat{c}^2(x_1)\frac{\partial^2 q}{\partial x^2} + \alpha\epsilon\hat{c}(x_1)\hat{c}'(x_1)\frac{\partial q}{\partial x} - \frac{\alpha\epsilon^2}{1+\epsilon x}\hat{c}(x_1)\hat{c}'(x_1)q,$$

where $q(x,\tau) = Q(r,t)/Q_0$, and a primes denotes differentiation w.r.t. argument.

Slowly-varying plane wave ansatz

- Can skip the introduction of a "short" time $\tau_1 = \epsilon\tau$ and proceed directly to the ansatz (similar to WKB),

$$q(x,\tau) = A(x_1)\exp[\mathrm{i}\tau - \mathrm{i}k(x_1)x].$$

- Under this ansatz, derivatives are, within $\mathcal{O}(\epsilon^2)$,

$$\begin{aligned}
\frac{\partial^2 q}{\partial \tau^2} &= -A\exp(\mathrm{i}\tau - \mathrm{i}kx),\\
\frac{\partial q}{\partial x} &= \left(\epsilon A' - \mathrm{i}kA - \mathrm{i}x\epsilon k' A\right)\exp(\mathrm{i}\tau - \mathrm{i}kx),\\
\frac{\partial^2 q}{\partial x^2} &= \left[-k^2 A - 2\mathrm{i}\epsilon(kA' + k'A) - 2\epsilon x k k' A\right]\exp(\mathrm{i}\tau - \mathrm{i}kx).
\end{aligned}$$

- Finally,

$$\left(1 - \hat{c}^2 k^2 - 2\epsilon x \hat{c}^2 k k'\right) A - 2\mathrm{i}\epsilon\hat{c}^2\left(kA' + k'A\right) - \alpha\mathrm{i}\epsilon\hat{c}\hat{c}' kA = 0.$$

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Solving for wavenumber k and amplitude A

- Separating the real and imaginary parts of this last equation, we arrive at the system (a bit like Eikonal & transport eq. in WKB)

$$1 - \hat{c}^2 \left(x_1 k^2\right)' = 0,$$
$$\left(kA\right)' + \frac{\alpha \hat{c}'}{2\hat{c}} kA = 0.$$

- Can be integrated to yield

$$k^2(x_1) = 2 + \frac{1}{(1-\delta)x_1} + \frac{(1+x_1)^2}{(\delta - 1 - x_1)x_1} + \frac{4\delta}{x_1} \tanh^{-1}\left(\frac{x_1}{2 - 2\delta + x_1}\right).$$

- Similarly,

$$A(x_1) = \frac{1}{k(x_1)\hat{c}^{\alpha/2}(x_1)},$$

where $\hat{c}(x_1) = 1 - \delta/(1 + x_1)$ and $\alpha \in \{1, 2\}$.

Cumulative effect on the wave number

- For $\delta \ll 1$ (small deviation from a flat spacetime),

$$k^2(x_1) \simeq 1 + \frac{2\delta}{x_1} \ln(1 + x_1).$$

or

$$k(x_1) \simeq 1 + \frac{\delta}{x_1} \ln(1 + x_1).$$

Thus, $k(0) \simeq 1 + \delta$ and $k(\infty) = 1$ [redshift].

- Then, the relative change in the wavenumber, as measured by a distance observer at $x_1 = X_1 \gg 1$ from the emitting surface, is

$$\frac{k(X_1) - k(0)}{k(0)} \simeq -\delta \left[1 - \frac{\ln(1 + X_1)}{X_1} \right].$$

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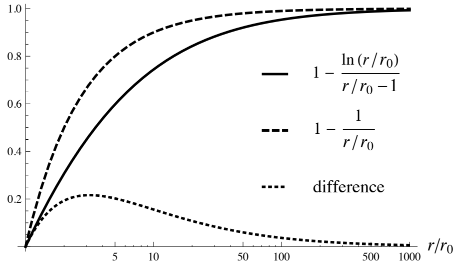
Cumulative effect on the wave frequency

- Frequency is $\omega(X_1) = \hat{c}(X_1)k(X_1) \simeq k(X_1)$ to the lowest order in δ so, in the original variables, within $\mathcal{O}(\delta^2)$,

$$\frac{\omega_{\rm rec} - \omega_{\rm em}}{\omega_{\rm em}} \simeq -\delta\left[1 - \frac{\ln(r/r_0)}{r/r_0 - 1}\right].$$

- Meanwhile, a quasi-constant light-speed approximation yields

$$\frac{\omega_{\rm rec} - \omega_{\rm em}}{\omega_{\rm em}} \simeq -\frac{k_0 c(r) - k_0 c(r_0)}{k_0 c(r_0)} = -\delta\left(1 - \frac{1}{r/r_0}\right).$$



The Doppler effect: Introduction

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"I love hearing that lonesome wail of the train whistle as the magnitude of the frequency of the wave changes due to the Doppler effect."

- First analyzed by Christian Doppler in the 19th century; now a cornerstone of classical and modern physics.

Classical Doppler

- Source moving with constant v towards fixed observer. Wave crests, initially emitted at wavelength λ (and arriving at period T) apart, start "bunching up":

$$\lambda_{\mathrm{D}} = \lambda - vT < \lambda.$$

- Measured wavelength is changed, hence heard frequency is as well:

$$f_{\mathrm{D}} \equiv \frac{c}{\lambda_{\mathrm{D}}} = \frac{c}{\lambda - vT} = \frac{1}{\lambda/c - v/cT} = \frac{1}{2\pi/(kc) - v/c(2\pi/\omega)}$$

or

$$\omega_{\mathrm{D}} \equiv 2\pi f_{\mathrm{D}} = \frac{2\pi}{2\pi/\omega - v/c(2\pi/\omega)} = \frac{\omega}{1 - v/c}.$$

- $f_{\mathrm{D}} > f$ is the frequency of the train whistle the cowboy hears when the train is approaching.

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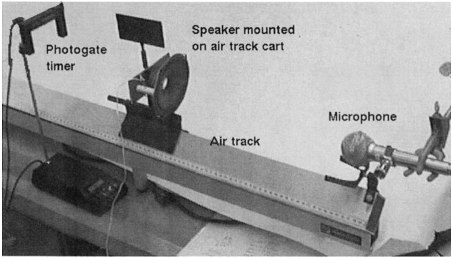
Accelerated Doppler: Non-constant v

- Simple experiment using a loudspeaker on an air track



(Bensky & Frey, *Am. J. Phys.*, 2001); see also (Azooz, *Am. J. Phys.*, 2007) suggest that

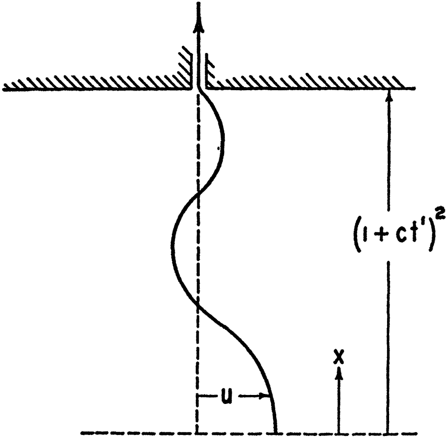
$$\omega_{\mathrm{D}}(t) = \frac{\omega}{1 - \dot{X}(t)/c}$$

holds for non-constant source velocity; position $X(t)$, velocity $\dot{X}(t)$, acceleration $\ddot{X}(t) \neq 0$.

- Can we perform a calculation to justify this?

Motivation: George Carrier's "Spaghetti Problem"

- What are the eigenmodes of a uniformly accelerated shortening string?



(Carrier, *Am. Math. Monthly*, 1949); detailed analyses by (Balazs, 1961; Greenspan, 1963; Cooper, 1993; ...) in *J. Math. Anal. Appl.*

Model problem: A moving boundary IBVP

- Consider ($c_0 = const.$)

$$\frac{\partial^2 U}{\partial t^2} - c_0^2 \frac{\partial^2 U}{\partial x^2} = 0, \qquad X_s(t) < x < \infty, \quad 0 < t < \infty.$$

- BC (*accelerating* source emitting monochromatic harmonic waves):

$$U\big(X_s(t), t\big) = e^{i\omega_0 t}, \qquad X_s(t) := v_0 t + a\alpha(\Omega t), \qquad t > 0,$$

where α is a specified dimensionless fucnt. (nonlinear in its argument).

- Supplement with a Sommerfeld radiation condition,

$$\lim_{x\to\infty} \left(\frac{\partial U}{\partial x} + i\kappa U \right) = 0,$$

where κ is the spatial wave number; "+" ensures waves are outgoing at $x = \infty$. [Work with complex U, take Re at the end...]

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Reformulation on a fixed domain

- Moving frame coordinates are

$$\xi = x - X_s(t) \equiv x - v_0 t - a\alpha(\Omega t), \qquad t = t.$$

- Then,

$$\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial t^2} - 2(v_0 + a\Omega\alpha')\frac{\partial^2}{\partial\xi\partial t} - a\Omega^2\alpha''\frac{\partial}{\partial\xi} + (v_0 + a\Omega\alpha')^2\frac{\partial^2}{\partial\xi^2}, \quad \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial\xi^2},$$

- Letting $U(x,t) = \tilde{U}(\xi,t)$,

$$\frac{\partial^2\tilde{U}}{\partial t^2} - 2(v_0 + a\Omega\alpha')\frac{\partial^2\tilde{U}}{\partial\xi\partial t} - a\Omega^2\alpha''\frac{\partial\tilde{U}}{\partial\xi} - \left[c_0^2 - (v_0 + a\Omega\alpha')^2\right]\frac{\partial^2\tilde{U}}{\partial\xi^2} = 0,$$
$$0 < \xi < \infty, \quad 0 < t < \infty,$$

  which is a wave equation with variable coefficients.
- Keep in mind that hyperbolicity requires that

$$|\dot{X}_s(t)| \equiv |v_0 + a\Omega\alpha'(\Omega t)| < c_0.$$

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Nondimensionalization

- Dimensionless independent variables and dimensionless parameters:

$$\tau = t/(\omega_0^{-1}), \;\; \eta = \xi/(c_0/\omega_0), \;\; \beta := v_0/c_0, \;\; \delta := a\Omega/c_0, \;\; \epsilon := \Omega/\omega_0 \ll 1.$$

- Letting $\tilde{U}(\xi, t) = \hat{U}(\eta, \tau)$, we have

$$\frac{\partial^2 \hat{U}}{\partial \tau^2} - 2\big[\beta + \delta\alpha'(\epsilon\tau)\big]\frac{\partial^2 \hat{U}}{\partial\eta\partial\tau} - \epsilon\delta\alpha''(\epsilon\tau)\frac{\partial \hat{U}}{\partial \eta} - \Big\{1 - \big[\beta + \delta\alpha'(\epsilon\tau)\big]^2\Big\}\frac{\partial^2 \hat{U}}{\partial \eta^2} = 0,$$
$$0 < \eta < \infty, \quad 0 < \tau < \infty.$$

- Finally, the boundary and radiation condition become

$$\hat{U}(0,\tau) = \mathrm{e}^{\mathrm{i}\tau}, \qquad \lim_{\eta\to\infty}\left(\frac{\partial \hat{U}}{\partial \eta} + \mathrm{i}k\hat{U}\right) = 0,$$

where $k = \kappa c_0/\omega_0$ is the dimensionless wave number.

Setup and preliminaries

- Introduce a "fast" time $t_0 = \tau$, a "slow" time $t_1 = \epsilon\tau$, a "short" spatial coordinate $y_0 = \eta$ and a "long" spatial coordinate $y_1 = \epsilon\eta$.
- PDE becomes, note the slowly varying coefficients,

$$\frac{\partial^2 \hat{U}}{\partial \tau^2} - 2\tilde{\beta}(t_1)\frac{\partial^2 \hat{U}}{\partial \eta \partial \tau} - \epsilon\delta\alpha''(t_1)\frac{\partial \hat{U}}{\partial \eta} - \left[1 - \tilde{\beta}^2(t_1)\right]\frac{\partial^2 \hat{U}}{\partial \eta^2} = 0,$$

  where $\tilde{\beta}(t_1) := \beta + \delta\alpha'(t_1)$.
- Let $\hat{U}(\eta, \tau) = \mathcal{U}(y_0, y_1, t_0, t_1)$, partial derivatives transforming as

$$\frac{\partial^2 \hat{U}}{\partial \tau^2} = \frac{\partial^2 \mathcal{U}}{\partial t_0^2} + 2\epsilon\frac{\partial^2 \mathcal{U}}{\partial t_0 \partial t_1} + \epsilon^2\frac{\partial^2 \mathcal{U}}{\partial t_1^2},$$

$$\frac{\partial \hat{U}}{\partial \eta} = \frac{\partial \mathcal{U}}{\partial y_0} + \epsilon\frac{\partial \mathcal{U}}{\partial y_1}, \qquad \frac{\partial^2 \hat{U}}{\partial \eta^2} = \frac{\partial^2 \mathcal{U}}{\partial y_0^2} + 2\epsilon\frac{\partial^2 \mathcal{U}}{\partial y_0 \partial y_1} + \epsilon^2\frac{\partial^2 \mathcal{U}}{\partial y_1^2},$$

$$\frac{\partial^2 \hat{U}}{\partial \eta \partial \tau} = \frac{\partial^2 \mathcal{U}}{\partial y_0 \partial t_0} + \epsilon\frac{\partial^2 \mathcal{U}}{\partial y_0 \partial t_1} + \epsilon\frac{\partial^2 \mathcal{U}}{\partial y_1 \partial t_0} + \epsilon^2\frac{\partial^2 \mathcal{U}}{\partial y_1 \partial t_1}.$$

Setup and preliminaries (continued)

- Upon introducing these into the PDE, keeping only terms up to ϵ,

$$\frac{\partial^2\mathcal{U}}{\partial t_0^2} - 2\tilde{\beta}(t_1)\frac{\partial^2\mathcal{U}}{\partial y_0\partial t_0} - \left[1-\tilde{\beta}^2(t_1)\right]\frac{\partial^2\mathcal{U}}{\partial y_0^2} + 2\epsilon\frac{\partial^2\mathcal{U}}{\partial t_0\partial t_1} - 2\epsilon\tilde{\beta}(t_1)\frac{\partial^2\mathcal{U}}{\partial y_0\partial t_1}$$
$$-2\epsilon\tilde{\beta}(t_1)\frac{\partial^2\mathcal{U}}{\partial y_1\partial t_0} - 2\epsilon\left[1-\tilde{\beta}^2(t_1)\right]\frac{\partial^2\mathcal{U}}{\partial y_0\partial y_1} - \epsilon\delta\alpha''(t_1)\frac{\partial\mathcal{U}}{\partial y_0} + \mathcal{O}(\epsilon^2) = 0.$$

- We proceed in the usual manner by a regular expansion:

$$\mathcal{U}(y_0, y_1, t_0, t_1) = \mathcal{U}_0(y_0, y_1, t_0, t_1) + \epsilon\mathcal{U}_1(y_0, y_1, t_0, t_1) + \cdots .$$

- The (first) BC becomes

$$\mathcal{U}_0(0, 0, t_0, t_1) = \mathrm{e}^{\mathrm{i}t_0}, \qquad \mathcal{U}_j(0, 0, t_0, t_1) = 0 \quad (j > 0).$$

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The leading-order problem

- Then, at the leading order, we have

$$\mathcal{L}_0[\mathcal{U}_0] = 0, \qquad \mathcal{L}_0 := \frac{\partial^2}{\partial t_0^2} - 2\tilde{\beta}\frac{\partial^2}{\partial y_0 \partial t_0} - (1-\tilde{\beta}^2)\frac{\partial^2}{\partial y_0^2}.$$

N.B.: $\tilde{\beta}$ does *not* depend on the fast time t_0.

- Clearly, a solution of the form

$$\mathcal{U}_0(y_0, y_1, t_0, t_1) = \mathcal{A}_0(y_1, t_1)\mathrm{e}^{\mathrm{i}t_0 - \mathrm{i}ky_0 + \mathrm{i}\psi_0(y_1,t_1)}$$

exists provided that

$$-1 - 2\tilde{\beta}k + \left(1-\tilde{\beta}^2\right)k^2 = 0 \quad \Longrightarrow \quad k = \frac{\tilde{\beta}\pm 1}{1-\tilde{\beta}^2} = \frac{\pm 1}{1\mp\tilde{\beta}}.$$

- Must pick the upper sign in the expression for k ($\Rightarrow k = 1/(1-\tilde{\beta}) > 0$) to satisfy the radiation condition.

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The $\mathcal{O}(\epsilon)$ problem

- Continuing to $\mathcal{O}(\epsilon)$, we must now solve

$$\mathcal{L}_0[\mathcal{U}_1] = -2\frac{\partial^2\mathcal{U}_0}{\partial t_0\partial t_1} + 2\tilde{\beta}\frac{\partial^2\mathcal{U}_0}{\partial y_0\partial t_1} + 2\tilde{\beta}\frac{\partial^2\mathcal{U}_0}{\partial y_1\partial t_0} + 2\left(1-\tilde{\beta}^2\right)\frac{\partial^2\mathcal{U}_0}{\partial y_0\partial y_1} + \delta\alpha''\frac{\partial\mathcal{U}_0}{\partial y_0}.$$

- Denote the RHS as $\mathfrak{F}$, and evaluate it based on $\mathcal{U}_0$ from :

$$\mathfrak{F}(y_0, y_1, t_0, t_1) = \Bigg\{ -2\left(\mathrm{i}\frac{\partial\mathcal{A}_0}{\partial t_1} - \mathcal{A}_0\frac{\partial\psi_0}{\partial t_1}\right) + 2\tilde{\beta}\left(-\mathrm{i}k\frac{\partial\mathcal{A}_0}{\partial t_1} + k\mathcal{A}_0\frac{\partial\psi_0}{\partial t_1}\right)$$
$$+2\tilde{\beta}\left(\mathrm{i}\frac{\partial\mathcal{A}_0}{\partial y_1} - \mathcal{A}_0\frac{\partial\psi_0}{\partial y_1}\right) + 2\left(1-\tilde{\beta}^2\right)\left(-\mathrm{i}k\frac{\partial\mathcal{A}_0}{\partial y_1} + k\mathcal{A}_0\frac{\partial\psi_0}{\partial y_1}\right) - \mathrm{i}\delta\alpha'' k\mathcal{A}_0\Bigg\} e^{\mathrm{i}t_0 - \mathrm{i}ky_0 + \mathrm{i}\psi_0}.$$

- Of course such an RHS will produce secular terms because $e^{\mathrm{i}t_0 - \mathrm{i}ky_0 + \mathrm{i}\psi_0}$ is in the nullspace of $\mathcal{L}_0$.
- Therefore, choose $\mathcal{A}_0$ and ψ_0 so that $\mathfrak{F} \equiv 0$.

The $\mathcal{O}(\epsilon)$ problem (continued)

- Separate real and imaginary parts of stuff in $\{\cdots\}$, assuming $\mathcal{A}_0 \neq 0$,

$$(1+\tilde{\beta}k)\frac{\partial \psi_0}{\partial t_1} - \left[\tilde{\beta} - \left(1-\tilde{\beta}^2\right)k\right]\frac{\partial \psi_0}{\partial y_1} = 0,$$
$$(1+\tilde{\beta}k)\frac{\partial \mathcal{A}_0}{\partial t_1} - \left[\tilde{\beta} - \left(1-\tilde{\beta}^2\right)k\right]\frac{\partial \mathcal{A}_0}{\partial y_1} = -\tfrac{1}{2}\delta\alpha'' k\mathcal{A}_0,$$

  a system of two decoupled hyperbolic PDEs, one with a source.

- Proceed by the method of characteristics, to get a set of ODEs:

$$\frac{\mathrm{d}t_1}{\mathrm{d}s} = 1+\tilde{\beta}(t_1)k, \qquad \frac{\mathrm{d}y_1}{\mathrm{d}s} = -\left\{\tilde{\beta} - \left[1-\tilde{\beta}^2(t_1)\right]k\right\},$$
$$\frac{\mathrm{d}\overline{\psi}_0}{\mathrm{d}s} = 0, \qquad \frac{\mathrm{d}\overline{\mathcal{A}}_0}{\mathrm{d}s} = -\tfrac{1}{2}\delta\alpha''(t_1)k\overline{\mathcal{A}}_0,$$

  where $\psi_0(y_1,t_1) = \overline{\psi}_0(s)$ and $\mathcal{A}_0(y_1,t_1) = \overline{\mathcal{A}}_0(s)$.

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Finishing up the integrations and putting it all together...

- We eliminate the slow/fast temporal/spatial coords to get

$$\hat{U}(\eta,\tau) \sim \exp\left\{-\frac{\delta}{2}\left[\alpha'\big((1-\beta)\epsilon\tau - \delta\alpha(\epsilon\tau) + \epsilon\eta\big) - \alpha'\big((1-\beta)\epsilon\tau - \delta\alpha(\epsilon\tau)\big)\right]\right\} \times \exp\left\{\mathrm{i}\left[\tau - \frac{\eta}{1-\tilde{\beta}(\epsilon\tau)}\right]\right\}.$$

- Returning to the fixed frame, in dimensional vars,

$$U(x,t) \sim \exp\left\{-\frac{\delta}{2}\left[\alpha'\big((1-\beta)\Omega t + x\Omega/c - \beta\Omega t - 2\delta\alpha(\Omega t)\big) - \alpha'\big((1-\beta)\Omega t - \delta\alpha(\Omega t)\big)\right]\right\} \times \exp\left\{\mathrm{i}\frac{\omega}{1-\beta-\delta\alpha'(\Omega t)}\left[t - x/c - \delta\alpha'(\Omega t)t + \delta\Omega^{-1}\alpha(\Omega t)\right]\right\}.$$

- And, in the limit $\delta \to 0$ (no acceleration, $\dot{X}_{\mathrm{s}}(t)/c \to \beta = const.$),

$$U(x,t) \sim \exp\left\{\mathrm{i}\frac{\omega}{1-\beta}\left[t - x/c\right]\right\}.$$

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Interpreting the multiple-scales solution

Our plane wave from "Waves 101" is $\mathrm{AM}(x,t)\,\mathrm{Re}\left\{\mathrm{e}^{\mathrm{i}\mathrm{FM}(t)\omega[t-x/c+\phi(t)]}\right\}$.

1. The frequency modulation (FM) factor is

$$\mathrm{FM}(t) := \frac{\omega_{\mathrm{D}}(t)}{\omega} = \frac{1}{1-[\beta+\delta\alpha'(\Omega t)]}, \qquad \dot{X}_{\mathrm{s}}(t)/c \equiv \beta + \delta\alpha'(\Omega t),$$

2. The amplitude modulation (AM) factor is

$$\mathrm{AM}(x,t) := \exp\left\{ -\frac{\delta}{2}\Big[\alpha'\big((1-\beta)\Omega t + x\Omega/c - \beta\Omega t - 2\delta\alpha(\Omega t)\big) - \alpha'\big((1-\beta)\Omega t - \delta\alpha(\Omega t)\big)\Big]\right\}.$$

3. There's also the time-dependent phase shift $\phi(t) = -\delta\alpha'(\Omega t)t + \delta\Omega^{-1}\alpha(\Omega t)$.

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Example: Oscillating source

- For example,

$$\alpha(t) = \sin(t) \quad \Rightarrow \quad \alpha'(t) = \cos(t), \;\; \max_{t\geq 0} |\alpha'(t)| = 1, \;\; \alpha(0) = 0.$$

- Can compute $\mathrm{AM}(x,t)$ and $\mathrm{FM}(t)$ exactly.

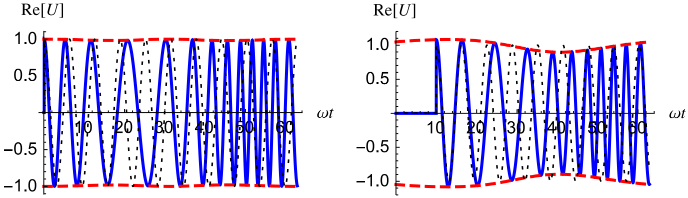
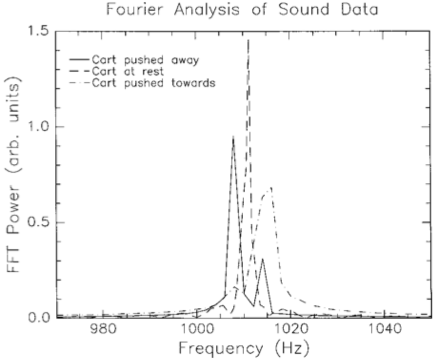


Figure: $\mathrm{Re}[\mathrm{AM}(x,t)e^{i\mathrm{FM}(t)\omega(t-x/c)}]$ (solid curves) for two receivers in the fixed frame; $\epsilon = 0.1$, $\beta = 0$, $\delta = 0.2$; $\omega x/c = 0.1$ (left) and $\omega x/c = 10$ (right). AM (dashed curves), the harmonic wave $\mathrm{Re}[e^{i\omega(t-x/c)}]$ (dotted curves).

An open problem

- Broadening of the spectral peak, enhanced secondary peak. Why?



(Bensky & Frey, *Am. J. Phys.*, 2001)

- Hypothesis: the variable phase shift $\phi(t)$ might have something to do with it. [Think power spectrum of $\sin(t)$ vs. $\sin(t + t^3)$...]

Summary

- First results on **cumulative effects due to variable phase speed** on plane wave propagation.
- Consistent **multiple scales asymptotic treatment**, free of secular terms.
- New **implications for accelerated Doppler shifts** (frequency and amplitude modulations).

- **Refs.:** I.C. Christov and C.I. Christov, "An improved formula for the frequency shift due to a variable phase speed," *J. Phys. A: Math. Theor.* **44** (2011) 112001, doi:10.1088/1751-8113/44/11/112001.

  I.C. Christov and C.I. Christov, "On mechanical waves and Doppler shifts from moving boundaries," *under review*.

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Thank you for your attention!

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