Dynamics of Polarization in Soliton Collisions in Coupled Nonlinear Schrodinger Equations

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The Big Picture

- Motivation:
  - Better understand how the dynamics of discrete quasi-particles arise from the dynamics of continuous coherent structures (e.g., solitons).
  - Develop simple models for a given property of a coherent structure in a (non-)integrable nonlinear evolution equation.
- Outline of the talk:
  1. Intro: Coupled nonlinear Schrödinger equations (CNLSE).
  2. Coarse-grain (variational) description of dynamics of polarization of CNLSE solitons.
  3. 2-soliton case, motivating example: sine-Gordon solitons as point particles with variable mass/size.
  4. Towards a simple model for the dynamics of polarization in CNLSE 2-soliton interactions?

Background

- Variational approximation (VA):
  - Sugiyama (*PTP*, 1979): ϕ^4 equation, TWS ansatz.
  - Anderson (*PRA*, 1983): nonlinear Schrödinger eq., Gaussian ansatz.
  - Ueda & Kath (*PRA*, 1990): coupled NLS eq's, $\approx$TWS ansatz.
  - Kaup & Malomed (*Phys. D*, 1995): variational principles and approximation for dissipative equations, ...
- Collective-variable/coordinate (CV) approach:
  - Rice (*PRB*, 1983): sine-Gordon equation, TWS ansatz.
  - Campbell, Schonfeld & Wingate (*Phys. D*, 1983): ϕ^4 equation, TWS ansatz, numerical study of kink disintegration.
  - Willis et al. (*PRA-B-E, Phys. D*, ca. 1989 to 1998): sG and ϕ^4 eq's, "Hamiltonian projection-operator approach," ...
- Coarse-grain description:
  - (I.C.C. & C.I.C., *PLA*, 2008, *DCDS*, 2009): apply VA using appropriately chosen CVs to study physics of interacting nonlinear waves as (quasi-)particles (QPs).

Coupled Nonlinear Schrödinger Equations

- Model eq's for light propagation in isotropic Kerr media:

$$i\psi_t + \beta\psi_{xx} + \left[\alpha_1|\psi|^2 + (\alpha_1 + 2\alpha_2)|\phi|^2\right]\psi = 0,$$
$$i\phi_t + \underbrace{\beta\phi_{xx}}_{\text{dispersion}} + \left[\underbrace{\alpha_1|\phi|^2}_{\text{self-focusing}} + (\alpha_1 + \underbrace{2\alpha_2)|\psi|^2}_{\text{cross-modulation}}\right]\phi = 0,$$

- Manakov (*JETP*, 1974) showed system is integrable only for $\alpha_2 = 0$.
- Solution ansatz based on a Manakov-type 1-soliton solution:

$$\psi(x,t;v,n_\psi,X,\delta_\psi) = A_\psi \operatorname{sech}[b_\psi(x-X)] \exp\left\{i\left[\frac{v}{2\beta}x + n_\psi t + \delta_\psi\right]\right\},$$
$$\phi(x,t;v,n_\phi,X,\delta_\phi) = A_\phi \operatorname{sech}[b_\phi(x-X)] \exp\left\{i\left[\frac{v}{2\beta}x + n_\phi t + \delta_\phi\right]\right\}.$$

- Plug into PDEs to show that

$$X = x_0 + vt, \quad b^2_{\psi,\phi} = \beta^{-1}\left(n_{\psi,\phi} + \tfrac{1}{4}v^2\right) \quad \left[n_{\psi,\phi} > -\tfrac{1}{4}v^2\right].$$

Coarse-Grain (Variational) Description for CNLSE

1. For a two-wave field we can formally write

$$\begin{pmatrix}\psi\\ \phi\end{pmatrix} = \vec{\chi}[x - X_1(t), (\text{other CVs})_1] + \vec{\chi}[x - X_2(t), (\text{other CVs})_2]$$

$$\cancel{+\vec{\chi}_{\text{interaction}}[x - X_1(t), (\text{other CVs})_1, x - X_2(t), (\text{other CVs})_2]}.$$

2. Neglect $\vec{\chi}_{\text{interaction}}$ because the interaction is captured by CVs.
3. CNLSE is the condition for stationarity of the following Lagrangian:

$$L(\psi,\phi;t) = \int_{-\infty}^{+\infty} \frac{i}{2}(\psi_t\bar{\psi} - \bar{\psi}_t\psi) + \frac{i}{2}(\phi_t\bar{\phi} - \bar{\phi}_t\phi) - \beta(|\psi_x|^2 + |\phi_x|^2)$$

$$+ \frac{\alpha_1}{2}(|\psi|^2 + |\phi|^2)^2 + 2\alpha_2|\psi|^2|\phi|^2 \, dx.$$

$\Rightarrow$ Compute L based on the ansatz for $(\psi,\phi)^\top$.

4. Find the Euler–Lagrange equations for the CVs.

- Notes: $\vec{\chi}$ is a CNLSE 1-soliton; can show $\|\vec{\chi}_{\text{interaction}}\| \ll \|\vec{\chi}\|$ *a posteriori*; initially well-separated QPs: $|X_1(0) - X_2(0)| \gg 1$.

Variational Description of 1 CNLSE QP (I)

- Suppose $n_\psi \neq n_\phi$; as an approximation take A_ψ and A_ϕ as CVs.
- What are the possible types of soliton "polarizations"?
- Coarse-grained (discrete) Lagrangian can be calculated:

$$\mathbb{L}(X, A_\psi, A_\phi; t) = -\left(\frac{2n_\psi\beta + X\ddot{X}}{b_\psi\beta}\right)A_\psi^2 - \left(\frac{2n_\phi\beta + X\ddot{X}}{b_\phi\beta}\right)A_\phi^2 - \beta\frac{2b_\psi}{3}A_\psi^2 - \beta\frac{2b_\phi}{3}A_\phi^2 + \alpha_1\frac{2}{3b_\psi}A_\psi^4 + \alpha_1\frac{2}{3b_\phi}A_\phi^4 + (\alpha_1 + 2\alpha_2)\underbrace{\mathfrak{C}(n_\psi, n_\phi)}_{\text{a hard integral}}A_\psi^2 A_\phi^2.$$

- Corresponding Euler–Lagrange equations for the CVs are

$$\begin{aligned}\ddot{X} &= 0,\\ (-6n_\psi - 2\beta b_\psi^2)A_\psi + 4\alpha_1 A_\psi^3 + 3(\alpha_1 + 2\alpha_2)b_\psi\mathfrak{C}(n_\psi, n_\phi)A_\psi A_\phi^2 &= 0,\\ (-6n_\phi - 2\beta b_\phi^2)A_\phi + 4\alpha_1 A_\phi^3 + 3(\alpha_1 + 2\alpha_2)b_\phi\mathfrak{C}(n_\psi, n_\phi)A_\psi^2 A_\phi &= 0.\end{aligned}$$

- Can show either $A_\psi^2/k_\psi^2 + A_\phi^2/k_\phi^2 = const.$ ("elliptic" polarization).
- $\Rightarrow A_\psi^2 + A_\phi^2 = const.$ ("circular" polarization) if $n_\psi = n_\phi$.
- Or, either $A_\psi = 0$ or $A_\phi = 0$ ("linear" polarization).

Variational Description of 1 CNLSE QP (II)

- For circular polarization: $n_\psi = n_\phi = n$, parametrize amplitudes as $A_\psi = A\cos\theta$, $A_\phi = A\sin\theta$.
- Coarse-grained (discrete) Lagrangian can be calculated:

$$\mathbb{L}(X,\theta;t) = -\frac{A^2}{b\beta}X\ddot{X} + \alpha_2\frac{4A^4}{3b}\sin^2(2\theta).$$

- Corresponding Euler–Lagrange equations for the CVs are

$$\frac{\delta\mathbb{L}}{\delta X} = -\frac{A^2}{b\beta}\ddot{X} = 0, \qquad \frac{\delta\mathbb{L}}{\delta\theta} = \alpha_2\frac{4A^4}{3b}\sin(4\theta) = 0.$$

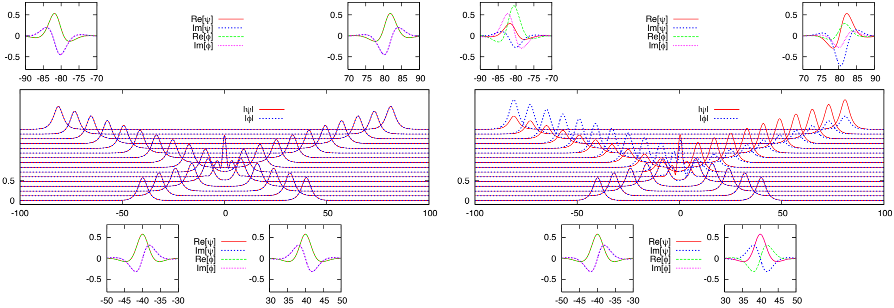
- Solutions of the latter are

$$X(t) = x_0 + v_0 t; \qquad \theta(t) = m\frac{\pi}{4}, \quad m\in\mathbb{Z}, \quad \text{or} \quad \alpha_2 = 0.$$

- For $\theta = 45^\circ$, Manakov's solution persists in the non-integrable case ($\alpha_2 \neq 0$); when $\alpha_2 = 0$ (integrable case) all θ are allowed.

Effect of Phase on 2-Soliton CNLSE Interactions ($\theta = 45^\circ$)

[Credit: Fig. 1 in Todorov & C.I.C. (*DCDS*, 2009); conservative FD scheme approach.]



(a) $\delta_l = \delta_r = 0^\circ$ (b) $\delta_l = 0^\circ$, $\delta_r = 90^\circ$

- No initial phase between components $\Rightarrow$ Manakov solitons persist.
- Initial phase $\Rightarrow$ different Manakov solitons emerge after collision.
  - Change occurs quickly during interaction: polarization shock

2 QPs with 2 CVs Each for CNLSE?

- Restrict to circular polarization; take X_is & θ_is as CVs.
- Result is an unwieldy discrete Lagrangian...

$$
\begin{aligned}
\mathbb{L} = & -\frac{A_1^2}{b_1\beta}X_1\ddot{X}_1 + \frac{8\alpha_2}{3b_1}A_1^4\sin^2\theta_1\cos^2\theta_1 - \frac{A_2^2}{b_2\beta}X_2\ddot{X}_2 + \frac{8\alpha_2}{3b_2}A_2^4\sin^2\theta_2\cos^2\theta_2 \\
& - \int_{-\infty}^{+\infty} dx\,\Re\left[\left\{ i\frac{\dot{v}_1+\dot{v}_2}{2\beta}x + i(n_1+n_2) + b_1\dot{X}_1\tanh[b_1(x-X_1)] - b_2\dot{X}_2\tanh[b_2(x-X_2)]\right\}(\Psi_1\bar{\Psi}_2 + \Phi_1\bar{\Phi}_2)\right] \\
& \qquad - \Re\left[\left\{\dot{\theta}_1\cot\theta_1 - \dot{\theta}_2\cot\theta_2\right\}\Psi_1\bar{\Psi}_2\right] - \Re\left[\left\{-\dot{\theta}_1\tan\theta_1 + \dot{\theta}_2\tan\theta_2\right\}\Phi_1\bar{\Phi}_2\right] \\
& - \int_{-\infty}^{+\infty} dx\,\Re\Big[\left\{ib_2\dot{X}_1\tanh[b_2(x-X_2)] - ib_1\dot{X}_2\tanh[b_1(x-X_1)] + 2\beta b_1 b_2\tanh[b_1(x-X_1)]\tanh[b_2(x-X_2)]\right\} \\
& \qquad \times(\Psi_1\bar{\Psi}_2 + \Phi_1\bar{\Phi}_2)\Big] \\
& + \frac{\alpha_1}{2}\int_{-\infty}^{+\infty} dx \times\Big\{2\Re[(\Psi_1\bar{\Psi}_2)^2] + 2\Re[(\Phi_1\bar{\Phi}_2)^2] + 4\left(|\Psi_1|^2 + |\Phi_1|^2 + |\Psi_2|^2 + |\Phi_2|^2\right)\Re[\Psi_1\bar{\Psi}_2] \\
& \qquad + 4\left(|\Psi_1|^2 + |\Phi_1|^2 + |\Psi_2|^2 + |\Phi_2|^2\right)\Re[\Phi_1\bar{\Phi}_2] + 4|\Psi_1|^2|\Psi_2|^2 + 2|\Psi_2|^2|\Phi_1|^2 \\
& \qquad + 2|\Psi_1|^2|\Phi_2|^2 + 4|\Phi_1|^2|\Phi_2|^2 + 4\Re[\Psi_1\bar{\Psi}_2\Phi_1\bar{\Phi}_2] + 4\Re[\Psi_2\bar{\Psi}_1\Phi_1\bar{\Phi}_2]\Big\} \\
& + 2\alpha_2\int_{-\infty}^{+\infty} dx\Big\{2|\Psi_1|^2\Re[\Phi_1\bar{\Phi}_2] + |\Psi_2|^2|\Phi_1|^2 + |\Psi_1|^2|\Phi_2|^2 + 2|\Psi_2|^2\Re[\Phi_1\bar{\Phi}_2] \\
& \qquad + 2|\Phi_1|^2\Re[\Psi_1\bar{\Psi}_2] + 2|\Phi_2|^2\Re[\Psi_1\bar{\Psi}_2] + 2\Re[\Psi_1\bar{\Psi}_2\Phi_1\bar{\Phi}_2] + 2\Re[\Psi_1\bar{\Psi}_2\Phi_2\bar{\Phi}_1]\Big\}
\end{aligned}
$$

Coarse-Grain Description of sine-Gordon Solitons

- Perring & Skyrme's (*Nucl. Phys.*, 1962) model unified field equation

$$u_{tt} - u_{xx} = -\sin u,$$

which renders the following Lagrangian stationary:

$$L = \frac{1}{2}\int_{-\infty}^{+\infty} u_t^2 - u_x^2 + 2(\cos u - 1)\,\mathrm{d}x.$$

- It has a kink 1-soliton solution:

$$\Phi(\xi; a) = 4\arctan[\exp(a\xi)],$$

- For a two-wave field we can formally write

$$u(x,t) = \Phi[x - X_1(t); a_1(t)] + \Phi[x - X_2(t); a_2(t)] + \text{small}.$$

- Suppose $a_i(t) \equiv 1$, $\dot{X}_i$, $\ddot{X}_i \ll 1$. Then (I.C.C. & C.I.C., *PLA*, 2008),

$$\mathbb{L} = \tfrac{1}{2}\mathbb{M}_{11}\dot{X}_1^2 + \underbrace{\mathbb{M}_{12}(X_2 - X_1)\dot{X}_1\dot{X}_2}_{\text{cross-mass}} + \tfrac{1}{2}\mathbb{M}_{22}\dot{X}_2^2 - \underbrace{U(X_2 - X_1)}_{\text{interaction potential}}.$$

2 Quasi-Particles with 2 Collective Variables Each

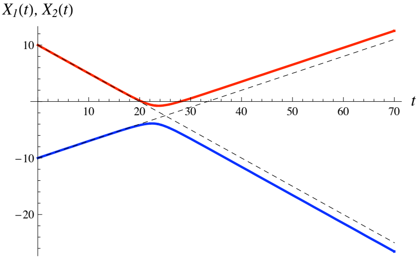
- But if $a_{1,2} \neq 1$ (I.C.C. & C.I.C., *DCDS*, 2009) we end up with

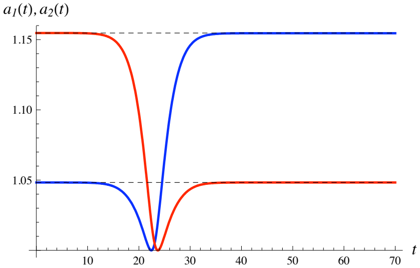
$$\begin{aligned}
\mathbb{L} = {} & 4a_1(\dot{X}_1^2 - 1) + \frac{\pi^2}{3}\frac{\dot{a}_1^2}{a_1^3} - \frac{4}{a_1} + 4a_2(\dot{X}_2^2 - 1) + \frac{\pi^2}{3}\frac{\dot{a}_2^2}{a_2^3} - \frac{4}{a_2} \\
& \mp 4a_1a_2(1 - \dot{X}_1\dot{X}_2)\int_{-\infty}^{+\infty} \operatorname{sech}[a_1(x - X_1)]\operatorname{sech}[a_2(x - X_2)]\,dx \\
& \pm 4\dot{a}_1\dot{a}_2 \int_{-\infty}^{+\infty} (x - X_1)(x - X_2)\operatorname{sech}[a_1(x - X_1)]\operatorname{sech}[a_2(x - X_2)]\,dx \\
& \mp 4\dot{a}_1 a_2\dot{X}_2 \int_{-\infty}^{+\infty} (x - X_1)\operatorname{sech}[a_1(x - X_1)]\operatorname{sech}[a_2(x - X_2)]\,dx \\
& \mp 4a_1\dot{a}_2\dot{X}_1 \int_{-\infty}^{+\infty} (x - X_2)\operatorname{sech}[a_1(x - X_1)]\operatorname{sech}[a_2(x - X_2)]\,dx \\
& \mp 4\int_{-\infty}^{+\infty} \frac{\sinh[a_1(x - X_1)]\sinh[a_2(x - X_2)] - 1}{\cosh^2[a_1(x - X_1)]\cosh^2[a_2(x - X_2)]}\,dx.
\end{aligned}$$

- Cannot evaluate these integrals analytically...
- Expand asymptotically for moderate phase speeds: $a_i = 1 - \epsilon^2\gamma_i$.

2 QPs with 2 CVs Each: Interaction, Mass Exchange

- Compute variational derivatives $\delta\mathbb{L}/\delta X_1$, $\delta\mathbb{L}/\delta a_1$, $\delta\mathbb{L}/\delta X_2$ and $\delta\mathbb{L}/\delta a_2$ to get highly nonlinear stiff system of 4 ODEs that is not resolved with respect to the highest order derivatives.
- Can solve by a semi-empirical fixed-point iteration scheme (I.C.C. & C.I.C., *DCDS*, 2009).





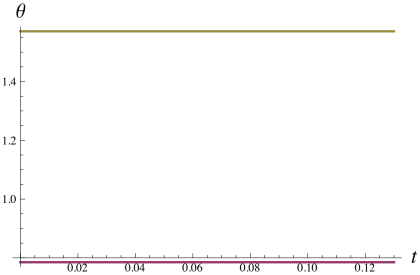
- QPs do not "pass through each other":
  - $\Rightarrow$ exchange of identity during interaction!

Simple Model for Polarization?

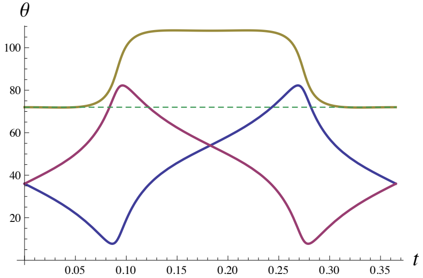
- *Ad hoc* modification of the discrete Lagrangian decouples X_is & θ_is:

$$\mathbb{L} \sim C_1 \cos(\theta_1 - \theta_2) - C_2(\dot{\theta}_1 + \dot{\theta}_2)\sin(\theta_1 - \theta_2) + \alpha_2 C_3 \cos(4\theta_1) + \alpha_2 C_4 \cos(4\theta_2) + \tfrac{1}{2}(\alpha_1+\alpha_2) C_5 \{\cos[2(\theta_1 - \theta_2)] - 3\cos[2(\theta_1 + \theta_2)]\} + 4\alpha_2 C_7 \cos(\theta_1 - \theta_2)\sin(\theta_1+\theta_2)^2.$$

- Find E–L equations for θ_1, θ_2 and integrate ODEs numerically.



(a) $\theta_1 = \theta_2 = 45^\circ$



(b) $\theta_1 = \theta_2 = 36^\circ$

- Consistent with C.I.C. & Todorov (*DCDS*, 2009) for $\delta_{\psi,1} = \delta_{\phi,1} = \delta_{\psi,2} = \delta_{\phi,2} = 0$.

Summary & conclusions

- What we have done:
  1. Coarse-grain description of a single CNLSE QP captures its polarization dynamics.
  2. For two CNLSE QPs, neither an exact nor an asymptotic evaluation of the discrete Lagrangian $\mathbb{L}$ appears possible.
     - ★ Approach to coarse-graining 2-QP interactions in sG appears inapplicable to CNLSE.
  3. *Ad hoc* model for decoupling polarization from trajectories has the correct phenomenology.

- What we have yet to do:
  1. Test the *ad hoc* model further to establish its utility.
  2. Include effects of phase $\delta_{\psi,\phi}$ within coarse-grain description.
  3. Explore parameter space of CNLSE, classify all types of polarization dynamics.

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