Eigenfunction expansions for sixth-order boundary value problems arising in elastic-plated thin-film dynamics

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General Fluid Dynamics: General II (R31.00004)
76th Annual Meeting of the American Physical Society Division of Fluid Dynamics
Washington, DC

November 20, 2023

Motivation for studying elastic-plated thin-film dynamics

credit: *Encyclopedia Britannica*

laccolith: an earthen dome created by magma flow

(see, also, Michaut *J. Geophys. Res.: Solid Earth*, 2011; Bunger & Cruden, *ibid*)

Also:

- growth of thin silicon oxide layers for semiconductors (see, *e.g.*, King, *SIAM J. Appl. Math.*, 1989),
- blister formation on skin (see, *e.g.*, Sulzberger *et al.*, *J. Invest. Dermatol.*, 1966; Juel *et al.*, *ARFM*, 2018).

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skin on cooling boiled milk

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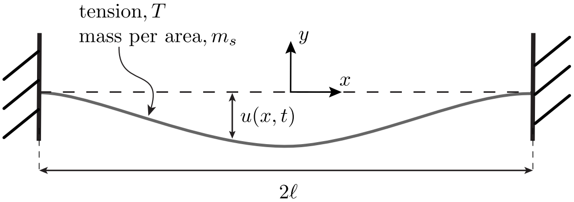
Features:

- elastic film over fluid on **confined** domain,
- has some bending resistance, little tensile strength.

Review: Second-order BVP – Stretching sheet

$$\begin{cases} -\dfrac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = \lambda^2\psi, \\ \psi(-1) = \psi(+1) = 0. \end{cases}$$



$$\begin{cases} m_s \dfrac{\partial^2 u}{\partial t^2} - T\dfrac{\partial^2 u}{\partial x^2} = 0, \\ u(-\ell) = u(+\ell) = 0. \end{cases}$$

- Linearly independent set of eigenfunctions is:

$$\left\{ \psi_m^s(x) = \sin(\lambda_m x), \qquad \lambda_m = m\pi. \right.$$

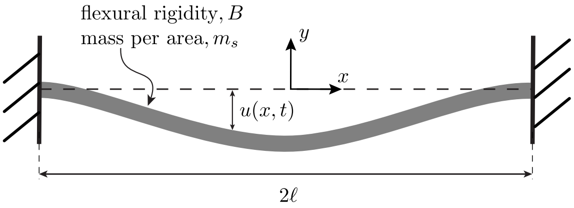
- $\{\psi_m^s\}$ form a **complete orthonormal set**, so

$$u(x) = \sum_{m=1}^{\infty} u_m^s \psi_m^s(x),$$

for any $u \in L^2[-1,+1]$.

Review: Fourth-order BVP – Bending sheet

$$\begin{cases} +\dfrac{\mathrm{d}^4\psi}{\mathrm{d}x^4} = \lambda^4\psi, \\ \psi(\pm 1) = \left.\dfrac{\mathrm{d}\psi}{\mathrm{d}x}\right|_{x=\pm 1} = 0. \end{cases}$$



- **Two** linearly independent sets of eigenfunctions:

$$\begin{cases} \psi_m^s(x) = \dfrac{1}{\sqrt{2}}\left[\dfrac{\sinh(\lambda_m^s x)}{\sinh(\lambda_m^s)} - \dfrac{\sin(\lambda_m^s x)}{\sin(\lambda_m^s)}\right], & \coth\lambda_m^s - \cot\lambda_m^s = 0, \\ \psi_m^c(x) = \dfrac{1}{\sqrt{2}}\left[\dfrac{\cosh(\lambda_m^c x)}{\cosh(\lambda_m^c)} - \dfrac{\cos(\lambda_m^c x)}{\cos(\lambda_m^c)}\right], & \tanh\lambda_m^c + \tan\lambda_m^c = 0. \end{cases}$$

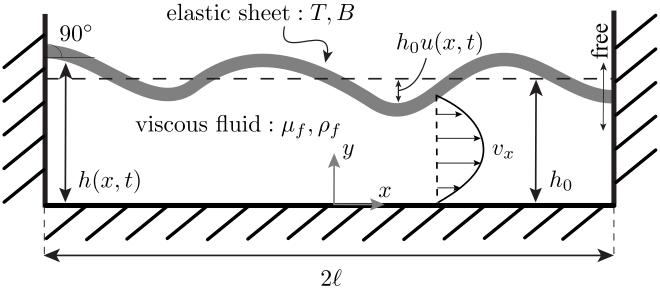
$$\begin{cases} m_s\dfrac{\partial^2 u}{\partial t^2} + B\dfrac{\partial^4 u}{\partial x^4} = 0, \\ u(\pm\ell) = \left.\dfrac{\partial u}{\partial x}\right|_{x=\pm\ell} = 0. \end{cases}$$

- $\{\psi_m^s, \psi_m^c\}$ form a **complete orthonormal set**, so

$$u(x) = \sum_{m=1}^{\infty} u_m^c\psi_m^c(x) + u_m^s\psi_m^s(x).$$

(Papanicolaou *et al.*, *IJNMF*, 2009; Chandrasekhar, *Hydrodynamic and Hydromagnetic Instability*, 1961; Lord Rayleigh, *Theory of Sound*, 1877)

Problem statement: confined elastic-plated thin-film dynamics



μ_f	:	fluid's dynamic viscosity [Pa s]	T	:	interface's elastic tension [N m^{-1}]
ρ_f	:	fluid's density [kg m^{-3}]	B	:	interface's elastic bending rigidity [Pa m^3]
$h(x,t)$	:	(total) height [m]	h_0	:	flat state [m]
$u(x,t)$	:	dimensionless displacement from the flat state, $h(x,t) = h_0[1+u(x,t)]$	2ℓ	:	width of trough [m]

Lubrication model

- For a slender film, $h \ll \ell$, neglecting body forces:

$$[x - \text{momentum}]: \quad 0 \approx -\frac{\partial p}{\partial x} + \mu_f \frac{\partial^2 v_x}{\partial y^2}, \qquad v_x(x,0,t) = v_x(x,h,t) = 0,$$

$$[y - \text{momentum}]: \quad 0 \approx -\frac{\partial p}{\partial y},$$

(see also, Oron *et al.*, *Rev. Mod. Phys.*, 1997; Hosoi & Mahadevan, *PRL*, 2004; Leal, *Advanced Transport Phenomena*, 2007)

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$$\Rightarrow v_x(x,y,t) \approx -\frac{1}{2\mu_f}\frac{\partial p}{\partial x} y[h(x,t) - y],$$

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$$[\text{continuity}] : \quad \frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\int_0^{h(x,t)} v_x \, \mathrm{d}y = \frac{\partial}{\partial x}\left[\frac{h^3}{12\mu_f}\frac{\partial p}{\partial x}\right],$$

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$$[\text{interface}]: \quad \Rightarrow p(x,t) \approx p_0 - \underbrace{T\frac{\partial^2 h}{\partial x^2}}_{\text{in-plane elastic tension}} + \underbrace{B\frac{\partial^4 h}{\partial x^4}}_{\text{out-of-plane bending}}.$$

(see also, Oron *et al.*, *Rev. Mod. Phys.*, 1997; Hosoi & Mahadevan, *PRL*, 2004; Leal, *Advanced Transport Phenomena*, 2007)

Governing partial differential equation

- Eliminating p, the lubrication model is a single nonlinear PDE for $h(x,t)$:

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left\{\frac{h^3}{12\mu_f}\left[-T\frac{\partial^3 h}{\partial x^3} + B\frac{\partial^5 h}{\partial x^5}\right]\right\}.$$

- Making dimensionless, we introduce.

$$\mathscr{T} = \frac{T}{B/\ell^2} \simeq \frac{\text{tension force}}{\text{bending force}},$$

- For $h = h_0(1+u)$ with $u \ll 1$, linearize to obtain the **governing PDE**:

$$\frac{\partial u}{\partial t} = -\mathscr{T}\frac{\partial^4 u}{\partial x^4} + \frac{\partial^6 u}{\partial x^6}.$$

The sixth-order IBVP for the elastic-plated thin film

☞ Non-pinned film in a closed trough, without tension ($\mathscr{T} = 0$)

- We consider symmetric domains $\Rightarrow$ same BCs at $x = \pm 1$.
  - Nonwetting film (contact angle at the wall is 90°) $\Rightarrow$ $\partial u/\partial x = 0$.
  - Point forces at the ends provided by wall $\Rightarrow$ $\partial^2 u/\partial x^2 = 0$ and $\partial^3 u/\partial x^3 \neq 0$.
  - And $\partial^5 u/\partial x^5 = 0$ to ensure flux $q = \cancel{\mathscr{T} \frac{\partial^3 u}{\partial x^3}} - \frac{\partial^5 u}{\partial x^5} = 0$.

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☞ Initial boundary value problem (IBVP) for $u = u(x,t)$

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \dfrac{\partial^6 u}{\partial x^6} = 0, & (x,t) \in (-1,1) \times \in (0,\infty), \\
\left.\dfrac{\partial u}{\partial x}\right|_{x=\pm 1} = \left.\dfrac{\partial^2 u}{\partial x^2}\right|_{x=\pm 1} = \left.\dfrac{\partial^5 u}{\partial x^5}\right|_{x=\pm 1} = 0, & t \in (0,\infty), \\
u(x,0) = u^0(x), & x \in (-1,1).
\end{cases}
$$

The associated sixth-order Sturm–Liouville EVP

$$\begin{cases} -\dfrac{\mathrm{d}^6\psi}{\mathrm{d}x^6} = \lambda^6\psi, \\ \left.\dfrac{\mathrm{d}\psi}{\mathrm{d}x}\right|_{x=\pm1} = \left.\dfrac{\mathrm{d}^2\psi}{\mathrm{d}x^2}\right|_{x=\pm1} = \left.\dfrac{\mathrm{d}^5\psi}{\mathrm{d}x^5}\right|_{x=\pm1} = 0. \end{cases}$$

- For these 3 BCs (out of 8 possible triplets), we can prove the EVP is **self-adjoint**.

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Theorem (Coddington & Levinson, Thm. 2.1)

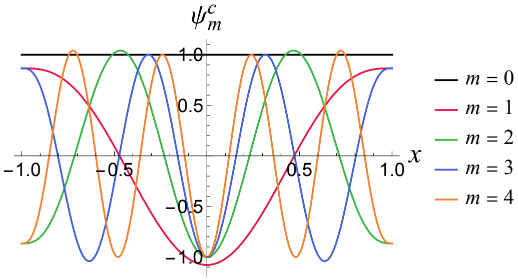
A self-adjoint EVP has a real, enumerable set of eigenvalues with corresponding ***distinct*** *and* ***mutually-orthogonal eigenfunctions****.*

Proposition (Direct application of Coddington & Levinson, Thm. 4.2)

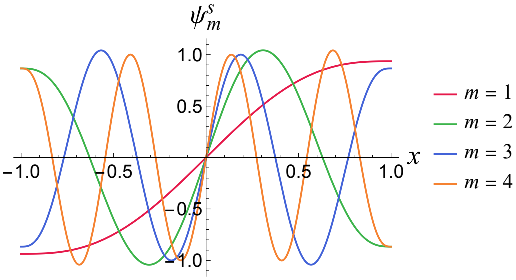
Solutions of this Sturm–Liouville EVP, supplemented by $\psi_0 = 1$, form a **complete orthonormal** set.

(see also general discussion of sixth- and higher-order EVPs by Greenberg & Marletta, *SIAM J. Numer. Anal.*, 1998)

The eigenfunctions can be explicitly constructed!



(a) Even eigenfunctions



(b) Odd eigenfunctions

$$\begin{cases} \psi_m^c(x) = c_m^c\left\{\frac{4\sin\lambda_m^c}{\cos\lambda_m^c - \cosh\sqrt{3}\,\lambda_m^c}\left[-\cos\frac{\lambda_m^c}{2}\sinh\frac{\sqrt{3}\,\lambda_m^c}{2}\sin\frac{\lambda_m^c}{2}x\sinh\frac{\sqrt{3}\,\lambda_m^c}{2}x + \sin\frac{\lambda_m^c}{2}\cosh\frac{\sqrt{3}\,\lambda_m^c}{2}\cos\frac{\lambda_m^c}{2}x\cosh\frac{\sqrt{3}\,\lambda_m^c}{2}x\right] + \cos\lambda_m^c x\right\}, \\ \qquad \cos 2\lambda_m^c + \sqrt{3}\sin\lambda_m^c\sinh\sqrt{3}\,\lambda_m^c - \cos\lambda_m^c\cosh\sqrt{3}\,\lambda_m^c = 0, \qquad \lambda_m^c \sim (m+1/6)\pi, \quad m\to\infty; \\ \psi_m^s(x) = c_m^s\left\{\frac{4\cos\lambda_m^s}{\cos\lambda_m^s + \cosh(\sqrt{3}\,\lambda_m^s)}\left[-\cos\frac{\lambda_m^s}{2}\cosh\frac{\sqrt{3}\,\lambda_m^s}{2}\sin\frac{\lambda_m^s}{2}x\cosh\frac{\sqrt{3}\,\lambda_m^s}{2}x + \sin\frac{\lambda_m^s}{2}\sinh\frac{\sqrt{3}\,\lambda_m^s}{2}\cos\frac{\lambda_m^s}{2}x\sinh\frac{\sqrt{3}\,\lambda_m^s}{2}x\right] + \sin(\lambda_m^s x)\right\}, \\ \qquad \sin 2\lambda_m^s + \sqrt{3}\cos\lambda_m^s\sinh\sqrt{3}\,\lambda_m^s + \sin\lambda_m^s\cosh\sqrt{3}\,\lambda_m^s = 0, \qquad \lambda_m^s \sim (m-1/3)\pi, \quad m\to\infty. \end{cases}$$

Model problem – Method of manufactured solution

$$\begin{cases} -\dfrac{d^6u}{dx^6} + 5544\dfrac{d^2u}{dx^2} + 199\,584\,u = \mathrm{rhs}(x), \\ \dfrac{du}{dx}\Big|_{x=\pm1} = \dfrac{d^2u}{dx^2}\Big|_{x=\pm1} = \dfrac{d^5u}{dx^5}\Big|_{x=\pm1} = 0, \\ \mathrm{rhs}(x) = 199\,584x^{12} - 465\,696x^{10} + 574\,560x^4 - 50\,1984x^2 + 147\,456. \end{cases}$$

- rhs is 'rigged' to produce the **even** exact solution $u_{\text{exact}}(x) = (x-1)^6(x+1)^6$.
- Galerkin expansion: $u \approx u_{\text{spectral}}(x) = \frac{1}{2}u_0\psi_0^c(x) + \sum_{n=1}^{M} u_n\psi_n^c(x)$ (no need for ψ^s).

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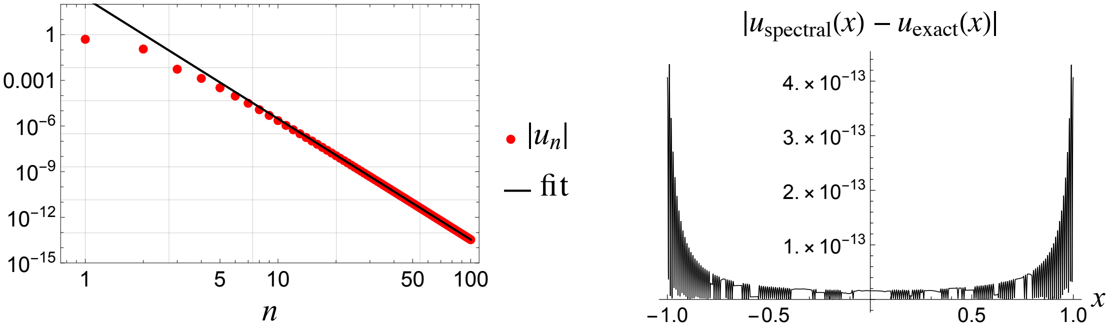
  - Need to solve a linear system for the coefficients

$$\sum_{m=1}^{M}\{(\lambda_n^c)^6\delta_{nm} + 5544\beta_{nm}^c + 199\,584\delta_{nm}\}u_m = \int_{-1}^{+1}\mathrm{rhs}(x)\psi_m^c(x)\,\mathrm{d}x, \qquad m = 1,\dots,M,$$

matrix is full because of $\beta_{nm}^c = \int_{-1}^{+1}\frac{\mathrm{d}^2\psi_n^c}{\mathrm{d}x^2}\psi_m^c\,\mathrm{d}x$ but **symmetric** positive definite, easily inverted; $u_0 = 2\,048/3\,003$.

Model problem – Numerical results

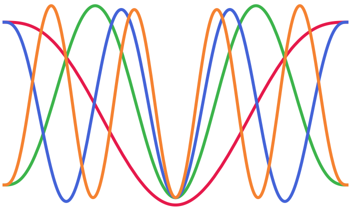


(a) (b)

Figure: (a) Convergence rate of u_{spectral}'s coefficients for the model problem; decay rate fit is $259n^{-7.94}$ (algebraic). (b) Error of u_{spectral} ($M = 100$ terms).

Summary

- *Explicitly* constructed complete orthonormal set of sixth-order eigenfunctions for problems elastic-plated thin-film flows.
- Demonstrated these eigenfunctions lead to *highly accurate* Galerkin expansions.
  - Also found *exact* formulæ for ψ'' and $\psi^{(iv)}$ in the basis.
- Next steps: construct Green's function & bring back external force(s), self-similar intermediate asymptotics?



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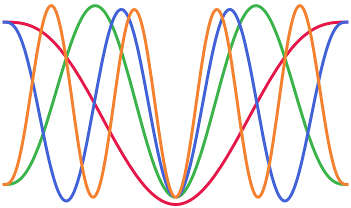
- Read our preprint at arXiv:2308.00673.



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