Resolving a paradox of anomalous scalings in the diffusion of granular materials

Ivan C. Christov and Howard A. Stone

Mechanical & Aerospace Engineering
Princeton University

Granular Flows I (G32.00010)
65th Annual Meeting of the APS DFD
Sand Diego, California

November 19, 2012
Diffusion of a “granular pulse” in a tumbler

- A band of colored particles in a monodisperse granular mixture diffuses axially as the container is rotated.
  

- Many regimes of flow in a tumbler...restrict to continuous flow.

- Thin surface shear layer $\Rightarrow$ velocity fluctuations $\Rightarrow$ lateral diffusion.

(Khan et al., *New J. Phys.* 2011)
Anomalous macroscopic vs. normal microscopic scalings

- **Experimental** concentration profiles of an axially diffusing pulse of dyed black salt grains in white salt grains: (Khan & Morris, Phys. Rev. Lett. 2005)

![Experimental concentration profiles](image-a)

- Mean square displacement (from simulations) vs time for a bed containing 25% small particles by volume: (Third et al., Phys. Rev. E 2011)

![Mean square displacement](image-b)
Intermediate asymptotics of diffusion

- Consider the (dimensionless) diffusion equation

\[
\frac{\partial C}{\partial T} = \frac{\partial}{\partial Z} \left( D(C, Z, \cdots) \frac{\partial C}{\partial Z} \right)
\]

with no flux BCs, \( \frac{\partial C}{\partial Z} \big|_{Z=\pm 1} = 0 \), and given IC, \( C(Z, 0) = C_i(Z) \).

- \( D = 1 \): random walk with \( \langle (\Delta Z)^2 \rangle \propto t \). (Einstein, Ann. Phys. 1905)
  - \( C \)-dependent jump probability leads to \( D(C) \). (Boon & Lutsko, EPL 2007)

- Similarity solution (\( D = 1 \)) of the form

\[
C(Z, T) = T^{-1/2} \mathcal{C}(Z T^{-1/2})
\]

(Barenblatt, Scaling, Self-similarity, and Intermediate Asymptotics, 1996)
Optimal self-similar solution

- Solutions for arbitrary initial data can be well-approximated (for $D = 1$ and $T > T^*$) by the point-source similarity solution

$$C(Z, T) = \frac{1}{\sqrt{4\pi(T + T^*)}} \exp \left\{ -\frac{(Z + Z^*)^2}{4(T + T^*)} \right\}$$

provided that

$$Z^* = -\frac{M_1}{M_0}, \quad T^* = \frac{1}{2} \left[ \frac{M_2}{M_0} - \left( \frac{M_1}{M_0} \right)^2 \right],$$

and $M_k := \int_{-\infty}^{+\infty} C_i(Z) Z^k \, dZ$.


- Note: $Z^* = 0$ in this talk ($z$-symmetric step function IC only).
Time-dependent anomalous collapse exponents

- Data collapses under rescaling \( C(Z, T) \rightarrow C(Z/T^\alpha, T) T^\alpha \)

\[ \Rightarrow \text{we are balancing } T^\alpha \text{ with } \sqrt{T + T^*}, \text{ meaning} \]

\[ \alpha(T) \simeq \frac{\ln(T + T^*)}{2 \ln T} = \begin{cases} 
\frac{1}{\ln T} \left[ \ln T^* + \frac{T}{2 T^*} + O(T^2) \right], & T \rightarrow 0, \\
\frac{1}{2} + \frac{1}{\ln T} \left[ \frac{T^*}{2 T} + O(T^{-2}) \right], & T \rightarrow \infty.
\end{cases} \]

Concentration-dependent diffusivity in bidisperse mixtures

- Suppose $C$ is the concentration of larger particles, then $D(C) = 1 + (C - 1/2)$. (reinterpretation of Ristow & Nakagawa, Phys. Rev. E 1999)
- Seeking a self-similar solution $C(Z, T) = T^{-\alpha} \mathcal{C}(\zeta)$ with $\zeta = T^{-\alpha} Z$:

$$0 = \mathcal{C} + \zeta \mathcal{C}' + \frac{(2 + 1)}{2\alpha} T^{1-2\alpha} \mathcal{C}'' + \frac{1}{2\alpha} T^{1-3\alpha} (\mathcal{C}^2)'' .$$

- If $\alpha = 1/3$, then $T^{1-2\alpha} = T^{1/3} \ll 1$ for $T \ll 1$.
- If $\alpha = 1/2$, then $T^{1-3\alpha} = T^{-1/2} \ll 1$ for $T \gg 1$.

![Graphs showing early times and late times for different values of $\alpha$.]
Summary

● What we have done:
  ▶ Everyone can be right! (Proposed a resolution to the granular diffusion “paradox.”)
  ▶ Analytical foundations for anomalous exponents in Fickian diffusion.

● What we haven’t done:
  ▶ Find an asymptotic solution matching across both scaling regimes in the nonlinear case?
  ▶ Take into account the disparate flow regimes (thin shear layer on top of bulk in solid body rotation)?

Thank you for your attention!