On a difficulty in eigenfunction expansion solutions for the start-up of fluid flow

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Introduction: Start-up of fluid flow

- In 1851, G.G. Stokes considered the question of how momentum is transported in a fluid due to the impulsive motion of a boundary.
- For the unidirectional \( \mathbf{u}(x, t) = (u(y, t), 0, 0) \) so \( \mathbf{u} \cdot \nabla \mathbf{u} \equiv 0 \) and \( \nabla \cdot \mathbf{u} \equiv 0 \).
- The resulting BVP, known as Stokes’ 1\textsuperscript{st} problem is

\[
\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial^2 \mathbf{u}}{\partial y^2},
\]

\[
\begin{align*}
\mathbf{u}(y, 0) &= 0, \quad 0 < y < 1, \\
u(0, t) &= 1, \quad t > 0, \\
u(1, t) &= 0, \quad t > 0.
\end{align*}
\]

- N.B.: \( u, y \) and \( t \) were rescaled to make BVP dimensionless.
- Solution can be found by standard techniques:

\[
u(y, t) = (1 - y) - \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \frac{2}{n\pi} \sin(n\pi y), \quad t > 0.
\]
Introduction: Generalization to non-Newtonian fluids

- Example non-Newtonian (viscoelastic) rheology

\[ \sigma + \tau \frac{\partial \sigma}{\partial t} = \dot{\gamma} + \alpha \frac{\partial \dot{\gamma}}{\partial t}, \quad \dot{\gamma} \equiv \frac{\partial u}{\partial y}. \]

- Eliminating \( \sigma \) using conservation of momentum, \( \partial u/\partial t = \partial \sigma/\partial y \), Stokes’ 1\(^{st} \) problem becomes

\[
\left(1 + \tau \frac{\partial}{\partial t}\right) \frac{\partial u}{\partial t} = \frac{\partial}{\partial y} \left(1 + \alpha \frac{\partial}{\partial t}\right) \frac{\partial u}{\partial y}, \quad (y, t) \in (0, 1) \times (0, \infty),
\]

\[
\begin{align*}
    u(y, 0) &= \frac{\partial u}{\partial t}(y, 0) = 0, & 0 < y < 1, \\
    u(0, t) &= 1, & t > 0, \\
    u(1, t) &= 0, & t > 0.
\end{align*}
\]

- What is the transient solution to this third-order BVP?
Textbook approach using eigenfunction expansions

(Carslaw and Jaeger, 1959; Batchelor, 1967; Leal, 2007; Bruus, 2008)

1. Obtain steady-state solution (elliptic problem): \( u_{ss}(y) = 1 - y \).

2. Make BVP homogeneous via \( u(y, t) = v(y, t) + u_{ss}(y) \), now
   \( v(0, t) = v(1, t) = 0 \) and \( v(y, 0) = -u_{ss}(y) \), \( \frac{\partial v}{\partial t}(y, 0) = 0 \).

3. Solve homogeneous BVP via eigenfunction expansion,
   \( v(y, t) = \sum_n a_n(t)\psi_n(y) \), where
   \[
   \frac{d^2}{dy^2}\psi_n(y) = -\lambda_n\psi_n(y), \quad \psi_n(0) = \psi_n(1) = 0.
   \]

4. Solve ODE for \( a_n(t) \) using favorite method.
Textbook approach vs. Laplace transform & numerics

**Figure:** Dashed: textbook approach; Solid: Laplace transform in $t$; circles: numerical solutions.

Many such unphysical and incorrect solutions can be found in the literature:

A resolution to the apparent difficulty

1. Obtain steady-state solution: \( u_{ss}(y) = 1 - y \).

2. Make BVP homogeneous via \( u(y, t) = v(y, t) + H(t)u_{ss}(y) \), now \( v(0, t) = v(1, t) = 0 \) and \( v(y, 0) = 0, \ \frac{\partial v}{\partial t}(y, 0) = 0 \).

3. BVP is homogeneous but now includes forcing on the RHS:

\[
\left(1 + \tau \frac{\partial}{\partial t}\right) \frac{\partial v}{\partial t} = \frac{\partial}{\partial y} \left(1 + \alpha \frac{\partial}{\partial t}\right) \frac{\partial v}{\partial y} - [\delta(t) + \tau \delta'(t)]u_{ss}(y).
\]

4. Now, solve BVP via eigenfunction expansion, \( v(y, t) = \sum_n a_n(t)\psi_n(y) \).

≈ Duhamels principle: time-varying boundary condition “exchanged” for a time-varying source term!
Discussion

- The transformation \( u(y, t) = v(y, t) + u_{ss}(y) \)
  - alters the solution for \( t < 0 \), which is supposed to be \( u \equiv 0 \) (fluid at rest);
  - “condenses” the cumulative effects from \( t = 0 \) to \( t = \infty \) into an initial condition, namely \( v(y, 0) = -u_{ss}(y) \)...violation of causality (“no input before the output”).

- The transformation \( u(y, t) = v(y, t) + H(t)u_{ss}(y) \)
  - respects condition of rest prior to start-up;
  - replaces time-dependent boundary condition with time-dependent source, initial condition is still state of rest \( (u \equiv 0) \).

- Is there a more general principle at play here?
Summary

- Proposed the correct, causality-satisfying transformation to a homogeneous BVP for start-up problems:

\[ u \mapsto u + H(t) u_{\text{steady-state}}. \]

- Confirmed with independent Laplace-transform solution and numerical simulations.

- Bold proposal: revise how we teach this topic?


Thank you for your attention!