Dissipative acoustic solitons

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In Memoriam

Christo Ivanov Christov (1951 – 2012)
http://christov.metacontinuum.com
The big picture

- Motivation:
  - Effect of fluctuations upon mean flow of a compressible fluid?
    - Average PDE $\rightarrow$ coarse-grained eq. (RANS & LES in turbulence).
    - Average the variational principle $\rightarrow$ “nonlinearly dispersive” corrections.
  - Implications for acoustics models? (Canonical problems.)
  - Vanishing-viscosity or vanishing-dispersion limit from NS to Euler?

- Outline of the talk:
  1. Derivation of weakly-nonlinear Lagrangian-averaged Euler-$\alpha$ (LAE-$\alpha$) model.
  2. “Dispersive Taylor shocks” under the linearized eq. of motion (EoM).
  3. Energy analysis and exact traveling wave solutions (TWSs) of the nonlinear problem.
  4. Numerical simulations of TWS interactions using scheme with internal iterations. (Dissipative solitons.)
Background

- **$\alpha$-models** of turbulence (Foias, Holm, Titi, Guermond, et al., ca. 1999–2003).
  - Introduce *energy penalty* in NS to prevent creation of excitations below a minimal length scale, $\alpha$.

- For incompressible fluids, $\alpha$ can also represent:
  - the cell size in a *finite-scale* Lagrangian numerical simulation (Margolin, TRSA, 2009),
  - the first material modulus of a *second-grade* non-Newtonian fluids (formal analogy).

- For compressible fluids, $\alpha$ is the flow-dependent *small-scale amplitude fluctuations* over which the Lagrangian mean is taken.

- Bhat & Fetecau (*DCDS-B*, 2006): LAE-$\alpha$ model of compressible flow for perfect gases (i.e., $\varphi \propto \varrho^\gamma$).

- We follow Lesser & Seebass (*JFM*, 1968): *finite-amplitude*, aka *weakly-nonlinear*, approach to eq. of motion for general $\varphi(\varrho)$. 
Eqs. of motion for 1D homentropic flow of an inviscid fluid

- Mass, momentum and energy conservation:
  \[ \rho_t + (\rho u)_x = 0, \]
  \[ \rho (u_t + uu_x) + \rho_x = 0 \alpha^2 [\rho (u_{xt} + uu_{xx})]_x, \]
  \[ \eta_t = \eta_t + u \eta_x = 0, \]
  where \( u = \nabla \phi = (u(x, t), 0, 0). \)

- Close system w/ quadratic approximation to barotropic eq. of state
  \[ \rho - \rho_e = \rho_e c_e^2 \left[ \frac{\rho - \rho_e}{\rho_e} + (\beta - 1) \left( \frac{\rho - \rho_e}{\rho_e} \right)^2 \right], \]

- Introduce dimensionless variables/quantities
  \[ s = (\rho - \rho_e)/\rho_e, \quad u^\diamond = u/V, \quad \phi^\diamond = \phi/(LV), \]
  \[ x^\diamond = x/L, \quad t^\diamond = t(c_e/L), \quad \epsilon = V/c_e, \quad a = \alpha/L. \]
  Mach number  Knudsen number
Weakly-nonlinear theory and reduction to a scalar equation

To justify continuum equation with averaged fluctuations, \( a \ll 1 \) (small Knudsen number).

Assume weak nonlinearity \( \epsilon \ll 1 \) and w.l.o.g. \( a^2 = \mathcal{O}(\epsilon) \).

After some work...

\[
\phi_{xx} - \phi_{tt} + a^2 \phi_{ttxx} = \epsilon \left[ 2(\beta - 1)\phi_t \phi_{xx} + \partial_t (\phi_x)^2 \right],
\]

a new weakly-nonlinear dispersive inviscid acoustic equation.

Distinguished limits:

- \( a \to 0 \): recover straightforward weakly-nonlinear lossless acoustic wave equation (see I.C.C., C.I.C. & Jordan, *QJMAM*, 2007).
- \( \epsilon \to 0 \): recover van Wijngaarden equation for sound waves in bubbly liquids (also Love’s equation from classical elasticity).
### Linearization: Love’s equation

1. Set $\epsilon = 0$ to get $\phi_{xx} - \phi_{tt} + a^2 \phi_{ttxx} = 0$.

2. The pressure satisfies the same eq., consider the signaling IBVP:

   $\rho_{xx} - \rho_{tt} + a^2 \rho_{ttxx} = 0, \quad (x, t) \in (0, \infty) \times (0, \infty),$

   $p(0, t) = H(t), \quad p(\infty, t) = 0, \quad t \in (0, \infty),$

   $p(x, 0) = 0, \quad p_t(x, 0) = 0, \quad x \in (0, \infty).$

3. Solve by **Laplace transform** in $t$ to get

   $$p(x, t) = H(t) \left[ 1 - \frac{2}{\pi} \int_0^{1/a} \frac{\cos(\eta t)}{\eta} \sin \left( \frac{\eta x}{\sqrt{1 - a^2 \eta^2}} \right) d\eta \right].$$

4. Or, solve by **Fourier sine transform** in $x$ to get

   $$p(x, t) = H(t) \left[ 1 - \frac{2}{\pi} \int_0^{\infty} \frac{\sin(\varsigma x)}{\varsigma(1 + a^2 \varsigma^2)} \cos \left( \frac{\varsigma t}{\sqrt{1 + a^2 \varsigma^2}} \right) d\varsigma \right].$$

Solution to the signaling IBVP for Love’s equation

Figure: The solution profile $p$ vs. $x$ of IBVP (solid) compared to the Taylor shock profile $\frac{1}{2}\{1 - \tanh[(x - t)/(2a)]\}$ (dashed); $a = 0.03$.

Notice, EoM is linear yet

1. there’s strong dispersive effects even for $a \ll 1$,
2. there’s strong coherence of the wavefront.

...reminiscent of zero-dispersion limit of KdV eq. (e.g., Grava & Klein, CPAM, 2007).
Energy analysis

- Recall our EoM: \( \phi_{xx} - \phi_{tt} + a^2 \phi_{ttxx} = \epsilon \left[ 2(\beta - 1)\phi_t \phi_{xx} + \partial_t (\phi_x)^2 \right] \).
- Assume localized solution \( (\lim_{|x| \to \infty} \phi_x, \phi_t \to 0) \) and integrate over \( x \):
  \[
  \frac{dE}{dt} = -2\epsilon (\beta - 3/2) \int_{-\infty}^{+\infty} (\phi_t)^2 \phi_{xx} \, dx.
  \]
- Energy is given by
  \[
  E := \frac{1}{2} \int_{-\infty}^{+\infty} \phi_x^2 + (\phi_t)^2 + a^2 (\phi_{xt})^2 \, dx.
  \]
  - **Non-conservative** nature of EoM entirely due to the nonlinearity.
  - Sign of \( \frac{dE}{dt} \) is not definite; for \( \beta = \frac{3}{2} \), \( \frac{dE}{dt} = 0 \).
  - For \( \beta \neq \frac{3}{2} \), \( \frac{dE}{dt} = 0 \) only for certain distinguished solutions, i.e.,
Traveling wave solutions

- EoM invariant under $x \mapsto -x$, consider only right-traveling waves: $\phi(x, t) = F(\xi)$ with $\xi := x - ct$.

- EoM is reduced to an ODE for $f = F'$:

$$3a^2 c^2 (f')^2 + 3(1 - c^2)f^2 + 2\epsilon\beta cf^3 = 6K_1 f + K_2.$$  

- For supersonic wave ($c > 1$) with $K_1 = K_2 = 0$:

$$f(\xi) = A_0 \text{sech}^2 \left[ \frac{(\xi - \xi_0)\sqrt{c^2 - 1}}{2ac} \right], \quad A_0 = \frac{3(c^2 - 1)}{2c\epsilon\beta},$$

- In terms of the potential $\phi$:

$$F(\xi) = \frac{2acA_0}{\sqrt{c^2 - 1}} \tanh \left[ \frac{(\xi - \xi_0)\sqrt{c^2 - 1}}{2ac} \right],$$

which is a topological soliton or kink.
Finite-difference scheme with internal iterations

- Introduce auxiliary function $\psi := \phi_t$, EoM becomes

$$\phi_{tt} = \phi_{xx} + a^2 \phi_{ttxx} - \epsilon [2(\beta - 1)\psi \phi_{xx} + 2\phi_x \psi_x].$$

- Use three-time-level, linearly-implicit discretization:

$$\delta_t + \delta_t - \Phi_j^n = \delta_x + \delta_x - \left[\frac{1}{4}(\Phi_{j}^{n+1} + 2\Phi_j^n + \Phi_j^{n-1})\right] + a^2 \delta_t + \delta_t - \delta_x + \delta_x - \Phi_j^n$$

$$- \epsilon \left[2(\beta - 1)\Psi_j^n \delta_x + \delta_x - \Phi_j^n + 2\delta_{x0} \Phi_j^n \delta_{x0} \Psi_j^n\right],$$

where $\Phi_j^n \approx \phi(x_j, t^n)$ and $\Psi_j^n \approx \psi(x_j, t^n)$.

- **Internal iterations** (e.g., C.I.C. & Velarde, *IJBC*, 1994) on $k$, guess $\Psi_{j}^{n,0} = \delta_t - \Phi_j^n$, find $\Phi_j^{n+1}$ from difference eq., then $\Psi_{j}^{n,k+1} = \delta_{t0} \Phi_j^n$.

- Convergence criterion $\max_j |\Psi_{j}^{n,k+1} - \Psi_{j}^{n,k}| < 10^{-8} \max_j |\Psi_{j}^{n,k}|$ is met within 3 to 10 internal iterations.
Simulation of soliton–anti-soliton interaction, $\beta < 3/2$

Figure: $\beta = 1.2$, $c_{1,2} = \pm 1.5$, $\xi_{0,\{1,2\}} = \mp 30$ and $T = 40$: space-time plot (left) and comparison of the post-interaction solitons with a linear superposition of the exact shapes (right). Clearly, the two solitons collide and re-emerge, retaining their shape (“identity”), save for a phase shift and the emission of small-amplitude radiation.
Simulation of soliton–soliton interaction, $\beta < 3/2$

Figure: $\beta = 1.2$, $c_{1,2} = \pm 1.5$, $\xi_0, \{1,2\} = \mp 30$ and $T = 40$: space-time plot (left) and comparison of the post-interaction solitons with a linear superposition of the exact shapes (right). The right soliton disintegrates almost immediately, and so does the left one upon reaching the remains of the right soliton.
Simulation of soliton–soliton interaction, $\beta > 3/2$

Figure: $\beta = 2.35$, $c_{1,2} = \pm 1.5$, $\xi_{0,\{1,2\}} = \pm 30$ and $T = 1$: space-time plot (left) and comparison of the post-interaction solitons with a linear superposition of the exact shapes (right). The right soliton develops a “horn” that grows without bound.
Summary & conclusions

What we have found:

1. A new 1D weakly-nonlinear LAE-$\alpha$ model eq. valid for gases & compressible liquids.
2. Kink TWSs can retain their identity after a collision.
3. Kink TWS behave as dissipative solitons.
4. Unstable combinations of kinks that disintegrate even before impact.
5. Instability can be explosive. [Proof?]

What we have yet to do:

1. Investigate of the stability of the kink solutions. [Proof?]
2. Check whether $\exists$ ranges of the wave speed $c$ for which the soliton–anti-soliton interaction leads to a bound state (“breathers”).
3. Can other weakly-nonlinear models (Rassmusen, Sørensen, et al., 2008–2011) support dissipative solitons?
4. Add dissipation (e.g., viscosity, heat conduction, ...).
Selected Bibliography


