Intermediate Asymptotics of Axial Diffusion of Tumbled Granular Materials

Ivan C. Christov and Howard A. Stone

Mechanical & Aerospace Engineering
Princeton University

Particulate and Multiphase Flows II
Engineering Sciences and Fundamentals
AIChE 2012 Annual Meeting

October 30, 2012

Grant # DMS-1104047

The big picture

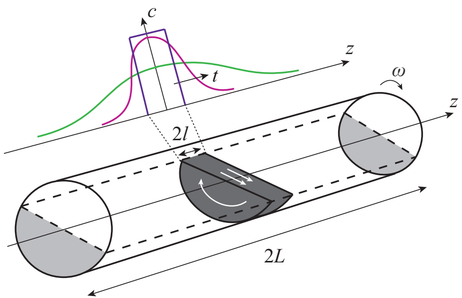
“**Can we develop a general theory of the dynamics of turbulent flows and the motion of granular materials?** *So far, such ‘nonequilibrium systems’ defy the tool kit of statistical mechanics, and the failure leaves a gaping hole in physics.*” — “So Much More to Know ...” (2005)

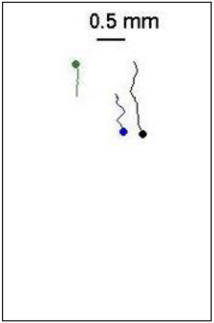
- Motivation:
  - Flowing granular materials — a complex system far from equilibrium.
    - ★ No “general theory” but simple models can be enlightening.
  - Diffusion, rheology, etc. in/of granular flow are not well understood.
- This talk:
  - Simple diffusion experiment with granular materials leads to anomalous scalings. Fundamentally new physics or not?

Diffusion of a "granular pulse" in a tumbler

- A band of colored particles in a monodisperse granular mixture diffuses axially as the container is rotated.

  (Lacey, *J. Appl. Chem.* 1954; Carley-Macauly & Donald, *Chem. Eng. Sci.* 1962)
- Many regimes of flow in a tumbler...restrict to continuous flow.
- Thin surface shear layer ⇒ velocity fluctuations ⇒ lateral diffusion.

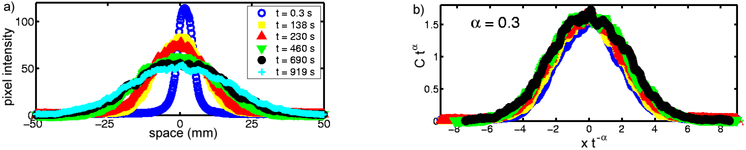




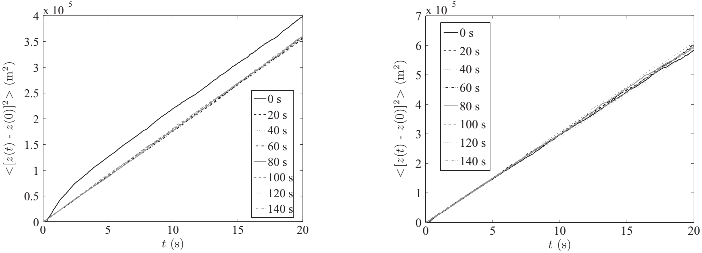
(Khan *et al.*, *New J. Phys.* 2011)

Anomalous macroscopic vs. normal microscopic scalings

- Experimental concentration profiles of an axially diffusing pulse of dyed black salt grains in white salt grains: (Khan & Morris, *Phys. Rev. Lett.* 2005)



- Mean square displacement (from simulations) vs time for a bed containing 25% small particles by volume: (Third *et al.*, *Phys. Rev. E* 2011)



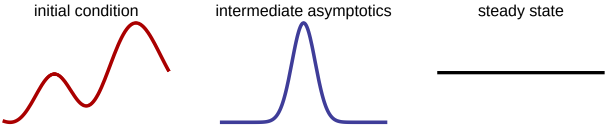
Intermediate asymptotics of diffusion

- Consider the (dimensionless) diffusion equation

$$\frac{\partial C}{\partial T} = \frac{\partial}{\partial Z}\left(\mathcal{D}(C, Z, \cdots)\frac{\partial C}{\partial Z}\right)$$

with no flux BCs, $\frac{\partial C}{\partial Z}\big|_{Z=\pm 1} = 0$, and given IC, $C(Z,0) = C_i(Z)$.

- $\mathcal{D} = 1$: random walk with $\langle(\Delta Z)^2\rangle \propto t$. (Einstein, *Ann. Phys.* 1905)
  - C-dependent jump probability leads to $\mathcal{D}(C)$. (Boon & Lutsko, *EPL* 2007)
- Similarity solution of the form $C(Z, T) = T^{-1/2}\mathfrak{C}(ZT^{-1/2})$.



(Barenblatt, *Scaling, Self-similarity, and Intermediate Asymptotics*, 1996)

Optimal self-similar solution

- Solutions for arbitrary initial data can be well-approximated (for $\mathcal{D} = 1$ and $T > T^*$) by the point-source similarity solution

$$C(Z,T) = \frac{1}{\sqrt{4\pi(T+T^*)}} \exp\left\{-\frac{(Z+Z^*)^2}{4(T+T^*)}\right\}$$

provided that

$$Z^* = -\frac{M_1}{M_0}, \quad T^* = \frac{1}{2}\left[\frac{M_2}{M_0} - \left(\frac{M_1}{M_0}\right)^2\right],$$

and $M_k := \int_{-\infty}^{+\infty} C_i(Z)\, Z^k \,\mathrm{d}Z$.
(Zel'dovich & Barenblatt, *Dokl. Akad. Nauk. SSSR* 1958)

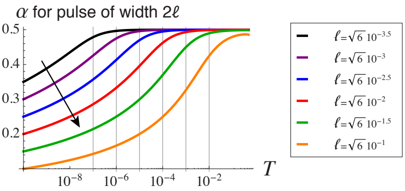
- Note: $Z^* = 0$ in this talk (z-symmetric step function IC only).

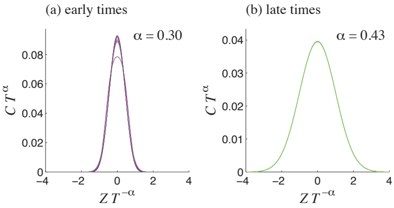
Time-dependent anomalous collapse exponents

- Data collapses under rescaling $C(Z,T) \to C(Z/T^{\alpha},T)T^{\alpha}$

  $\Rightarrow$ we are balancing T^{α} with $\sqrt{T+T^*}$, meaning

$$\alpha(T) \simeq \frac{\ln(T+T^*)}{2\ln T} = \begin{cases} \frac{1}{\ln T}\left[\ln T^* + \frac{T}{2T^*} + \mathcal{O}(T^2)\right], & T \to 0, \\ \frac{1}{2} + \frac{1}{\ln T}\left[\frac{T^*}{2T} + \mathcal{O}(T^{-2})\right], & T \to \infty. \end{cases}$$





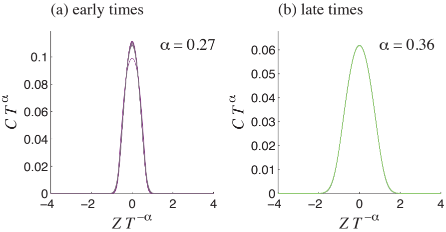
(Christov & Stone, *Proc. Natl Acad. Sci. USA* 2012)

Concentration-dependent diffusivity in bidisperse mixtures

- Suppose C is the concentration of larger particles, then $\mathcal{D}(C) = 1 + (C - 1/2)$. (reinterpretation of Ristow & Nakagawa, *Phys. Rev. E* 1999)
- Seeking a self-similar solution $C(Z,T) = T^{-\alpha}\mathfrak{C}(\zeta)$ with $\zeta = T^{-\alpha}Z$:

$$0 = \mathfrak{C} + \zeta\mathfrak{C}' + \underbrace{\frac{(2\mp 1)}{2\alpha}T^{1-2\alpha}\mathfrak{C}''}_{①} + \underbrace{\frac{1}{2\alpha}T^{1-3\alpha}\left(\mathfrak{C}^2\right)''}_{②}.$$

- If $\alpha = 1/3$, then $T^{1-2\alpha} = T^{1/3} \ll 1$ for $T \ll 1$.
- If $\alpha = 1/2$, then $T^{1-3\alpha} = T^{-1/2} \ll 1$ for $T \gg 1$.



Summary

- What we have done:
  - Everyone can be right! (Proposed a resolution to the granular diffusion "paradox.")
  - Analytical foundations for anomalous exponents in Fickian diffusion.
- What we haven't done:
  - Find an asymptotic solution matching across both scaling regimes in the nonlinear case?
  - Take into account the disparate flow regimes (thin shear layer on top of bulk in solid body rotation)?

Thank you for your attention!

Selected Bibliography

- Witelski & Bernoff, "Self-similar asymptotics for linear and nonlinear diffusion equations"
  Stud. Appl. Math. 100 (1998) 153–193.
- Khan & Morris "Subdiffusive Axial Transport of Granular Materials in a Long Drum Mixer"
  Phys. Rev. Lett. 94 (2005) 048002.
- Third, Scott & Müller, "Axial transport within bidisperse granular media in horizontal rotating cylinders"
  Phys. Rev. E 84 (2011) 041301.
- Khan et al., "High-speed measurement of axial grain transport in a rotating drum"
  New J. Phys. 13 (2011) 105005.
- Christov & Stone, "Resolving a paradox of anomalous scalings in the diffusion of granular materials"
  Proc. Natl Acad. Sci. USA 109 (2012) 16012–16017.