Structure of 3D chaotic transport in a tumbled granular flow in a sphere

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Granular flows

Movie credit: S.W. Meier

- Flowing granular matter is a complex system.
- No general theory; can understand mixing in terms of geometry and kinematics.
  - In the Lagrangian frame we study flows: $\mathrm{d}\vec{x}/\mathrm{d}t = \vec{v}(\vec{x}, t)$.
  - Dynamical systems framework: hyperbolic vs. elliptic periodic points, Poincaré sections, stable & unstable manifolds, etc.
  - Stirring by chaotic advection.
- Will study transport in a half-full sphere:
  - All “interesting” dynamics occur in a thin surface layer.
  - Motion (a) restricted to 2D surfaces or (b) fully-3D.
  - Goal: Explore kinematic flow structures in (a) and (b).

A granular flow that everyone can enjoy

Source: FoodNetwork's *Unwrapped* [Season 1, Episode 1 "Bubble Gum Unwrapped"].

Phenomenology of tumbled granular flow

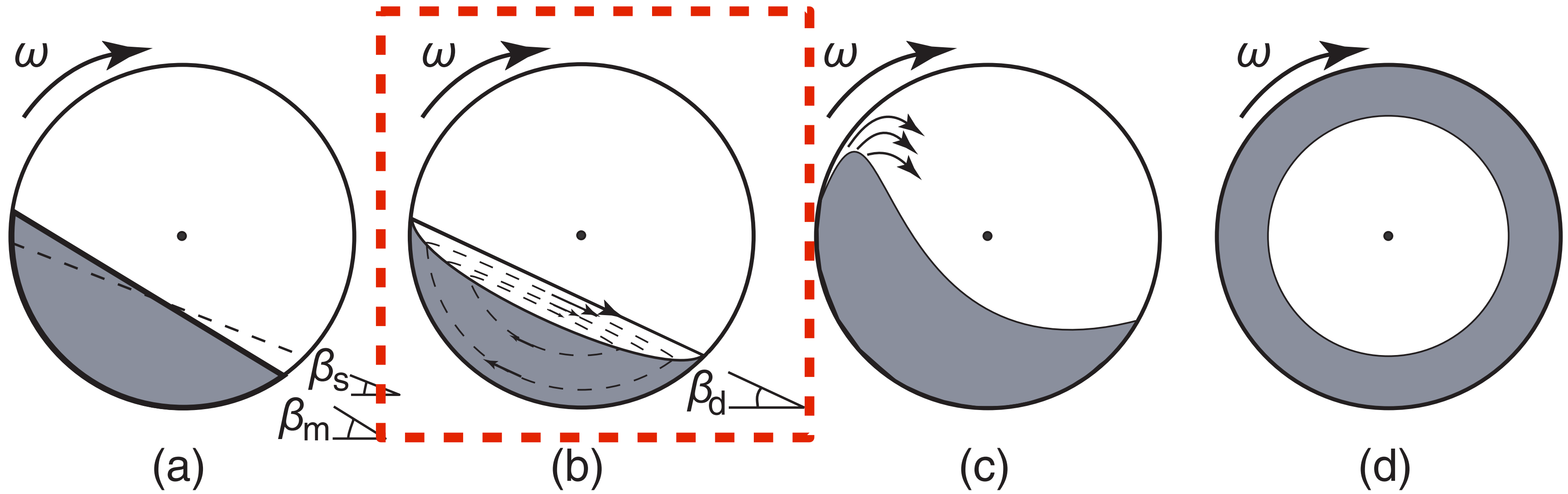


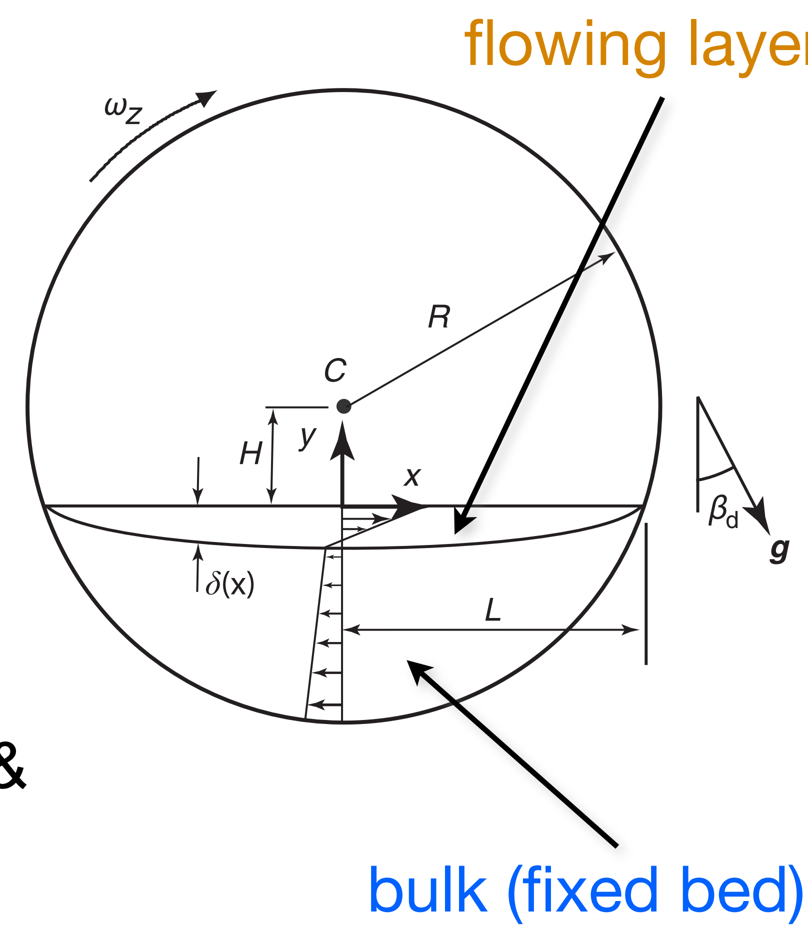
Figure 3. Illustration of flow regimes in tumblers: (a) avalanching; (b) rolling/continuous-flow/cascading; (c) cataracting; (d) centrifuging. [Meier, *et al.*, *Adv. Phys.* (2007)]

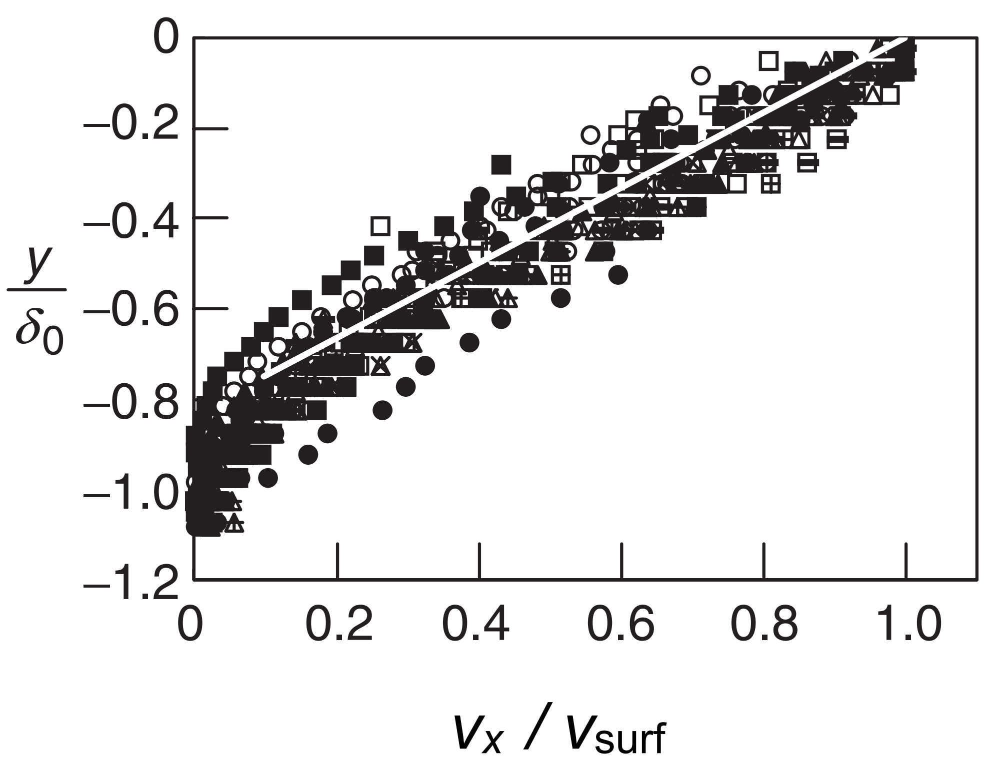
- Balance centrifugal and gravitational accel'ns: $\mathrm{Fr} = \omega^2 R/g$
- (a) $\mathrm{Fr} \lesssim 10^{-5}$
- (b) $10^{-4} \lesssim \mathrm{Fr} \lesssim 10^{-2}$ ($10^{-3} \lesssim \mathrm{Fr} \lesssim 10^{-1}$)
- (c) $10^{-1} \lesssim \mathrm{Fr} \lesssim 1$
- (d) $\mathrm{Fr} \gtrsim 1$

Continuum model of granular flow in the rolling regime

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \begin{cases} v_x(x(t), y(t), t), & y(t) > -\delta(x(t), t); \\ \omega_z y(t), & \text{otherwise.} \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = \begin{cases} v_y(x(t), y(t), t), & y(t) > -\delta(x(t), t); \\ -\omega_z x(t), & \text{otherwise.} \end{cases}$$





v_x as f'n of y for 27 dry & liquid granular mixtures

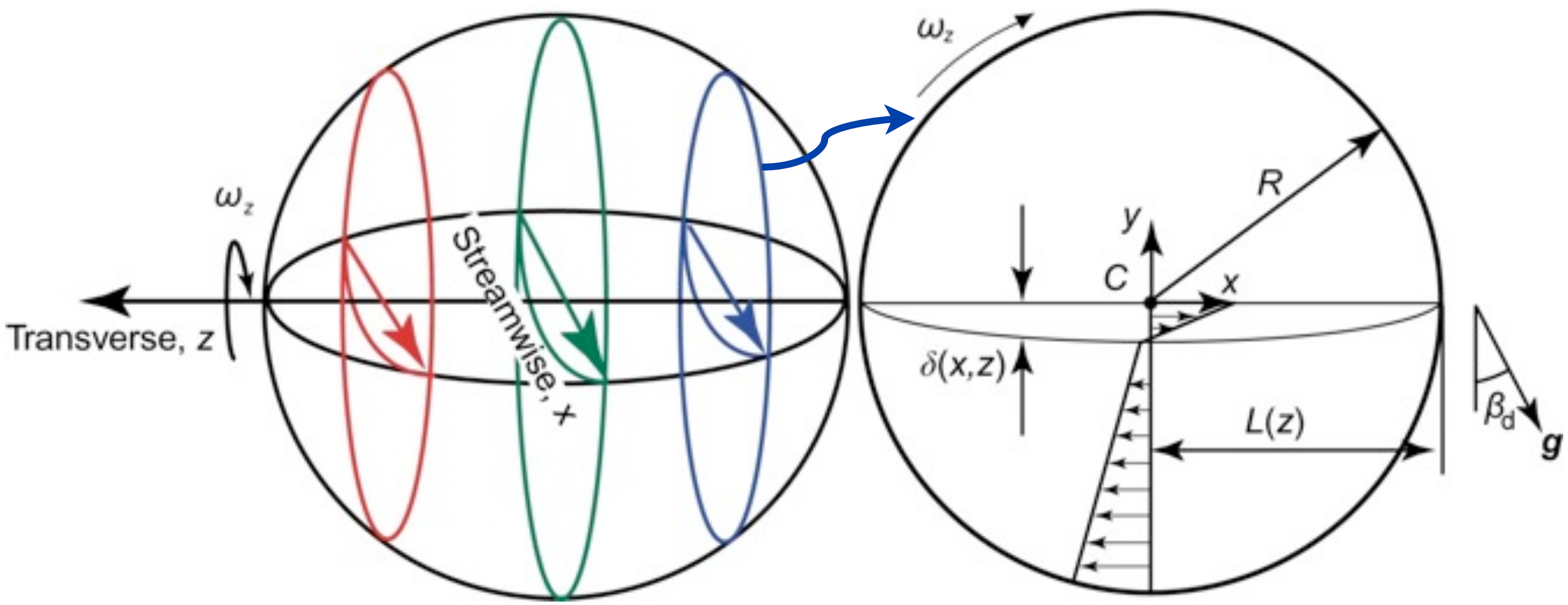
↓

$$\partial v_x / \partial y \approx const.$$

Simple shear in the flowing layer to a good approximation.

Jain *et al.*, *JFM* (2004); Meier *et al.*, *Adv. Phys.* (2007)

3D model is based on identical vertical 2D slices



$$\dot{x} = \begin{cases} \dot{\gamma}[\delta(x,z) + y], & y > -\delta(x,z); \\ -\omega_z y, & \text{otherwise.} \end{cases}$$

$$\dot{y} = \begin{cases} -\omega_z xy/\delta(x,z), & y > -\delta(x,z); \\ \omega_z x, & \text{otherwise.} \end{cases}$$

$$\dot{z} = 0$$

$$\delta(x,z) = \delta_0(z)\sqrt{1 - \frac{x^2}{L(z)^2}}$$

$$\delta_0(z) = \sqrt{\frac{|\omega_z|}{\dot{\gamma}}}L(z), \quad L(z) = \sqrt{R^2 - z^2}$$

Meier *et al.*, *Adv. Phys.* (2007); Sturman *et al.*, *JFM* (2008)

(and similarly for rotation about the *x*-axis...)

Rotation about a single axis is an integrable flow

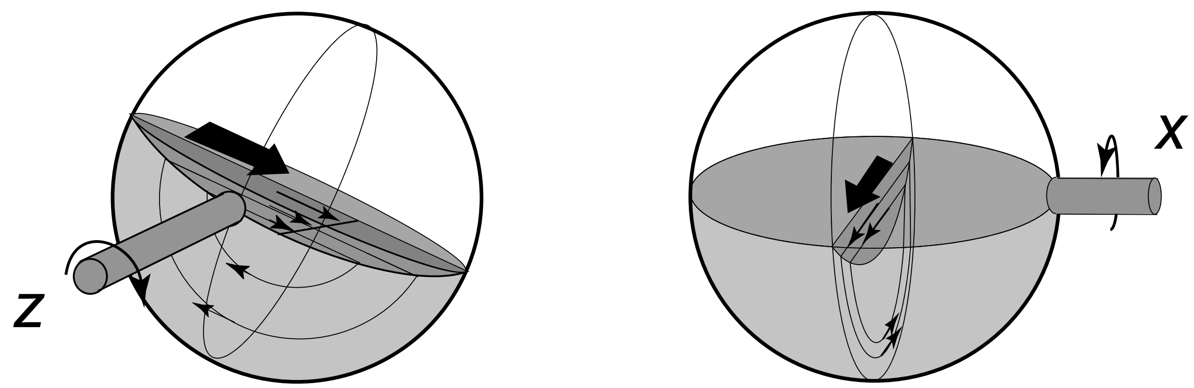
- Can calculate particle trajectories in 2D circular tumbler:

$$\begin{Bmatrix} x \\ y \end{Bmatrix}(t) = r_0 \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \left[-\omega_z t + \sin^{-1}\left(\frac{x_0}{r_0}\right) \right]. \qquad \text{(bulk)}$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix}(t) = \begin{Bmatrix} \sqrt{L^2+\kappa}\,\sin\left[\sqrt{\omega_z \dot{\gamma}_z}\,t + \sin^{-1}\left(\frac{x_0}{\sqrt{L^2+\kappa}}\right)\right] \\ -\sqrt{\frac{\omega_1}{\dot{\gamma}_1}}\sqrt{L^2 - x(t)^2} + \sqrt{\frac{\omega_1}{\dot{\gamma}_1}}\sqrt{L^2+\kappa-x(t)^2} \end{Bmatrix} \qquad \text{(flowing layer)}$$

- In the flowing layer, shear modifies the center and rate & sense of rotation.
- Composing two rotations gives a linked twist map with non-trivial dynamics [Sturman *et al*., *JFM* (2008)].

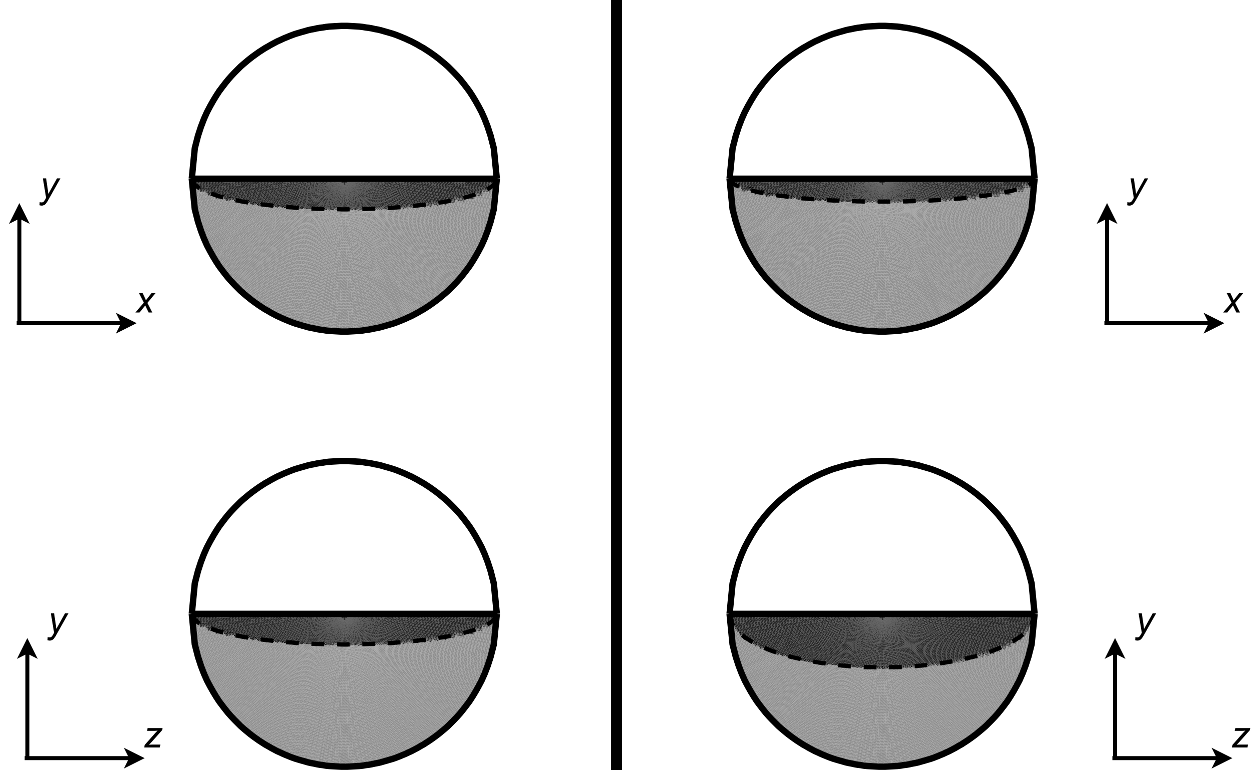
“Blinking” 3D tumbled granular flow

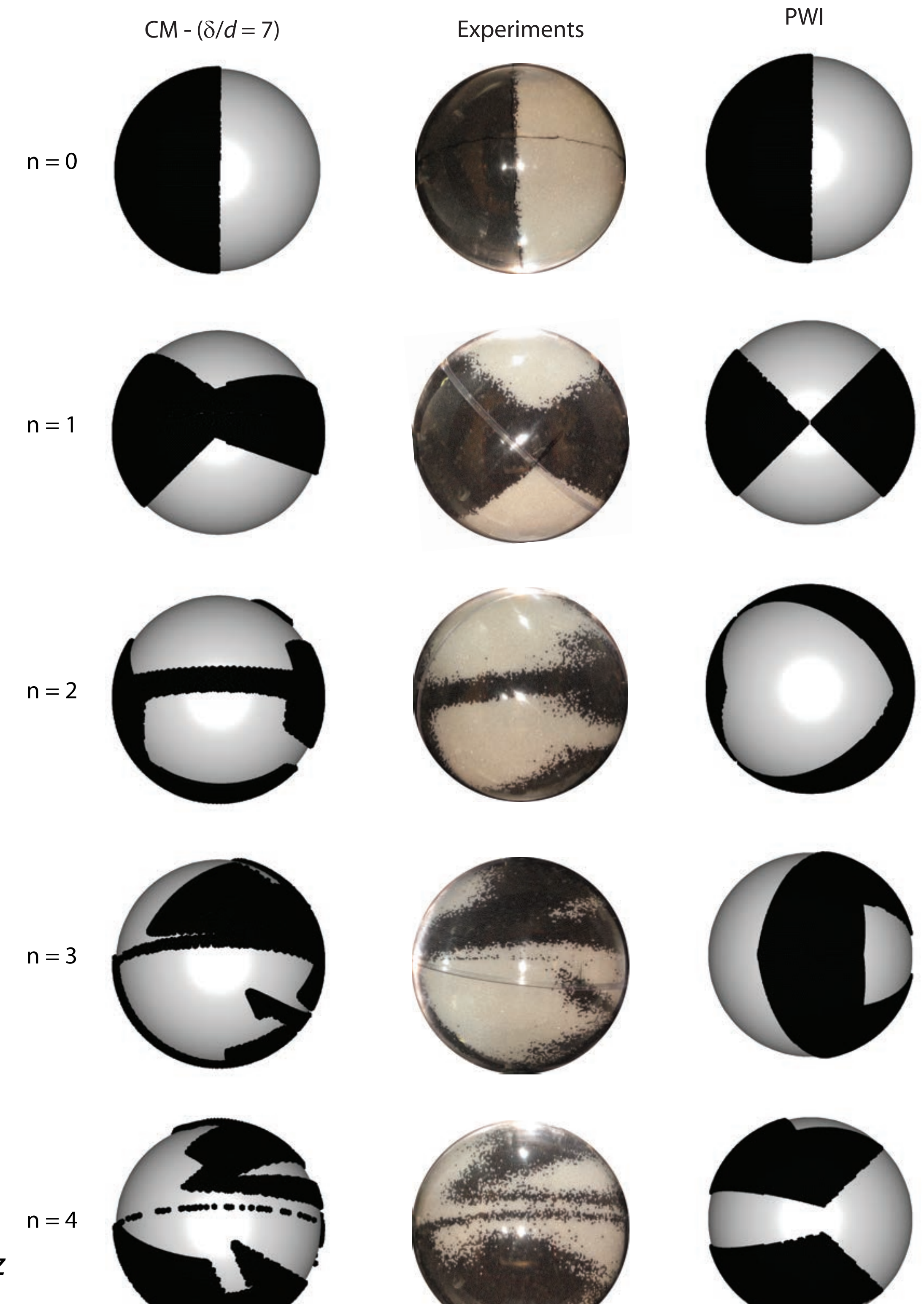


symmetric	non-symmetric
$\delta_{0,1} = \delta_{0,2}$ $(\omega_1 = \omega_2)$	$\delta_{0,1} \neq \delta_{0,2}$ $(\omega_1 \neq \omega_2)$

rotation 1

rotation 2





Juarez *et al.*, *EPL* (2010)

Particle motion: symmetric vs. non-symmetric case

Thm: A particle can change its distance r_0 from the origin iff the axis of rotation is switched while it is in the flowing layer and $\delta_{0,1} \neq \delta_{0,2}$:

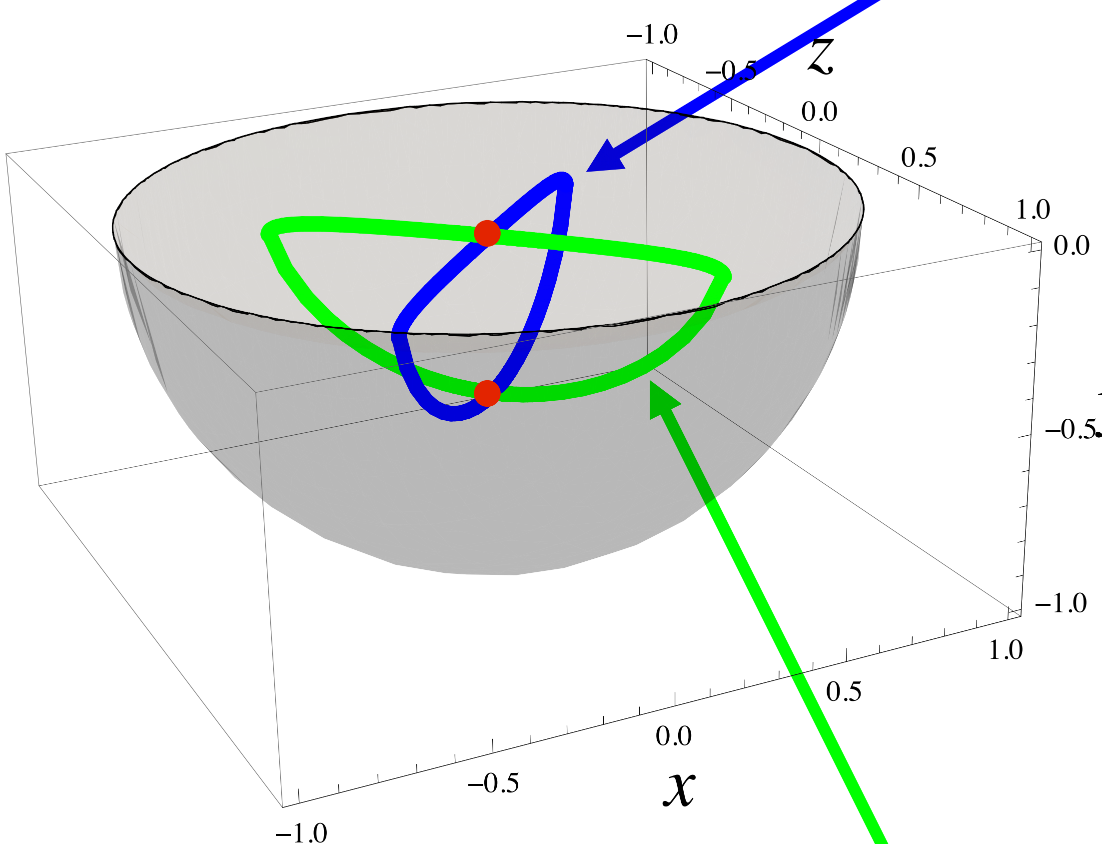
$$x_0^2 + y_0^2 + z_0^2 = r_0^2,$$
$$x_f^2 + y_f^2 + z_f^2 = 1 + (r_0^2 - 1)\frac{1-\delta_{0,2}^2}{1-\delta_{0,1}^2}.$$

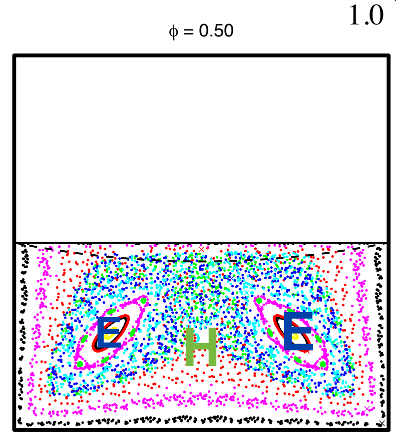
Cor 1: Switching the axis of rotation while a particle is in the bulk does not change its r_0.

Cor 2: For $\delta_{0,1} = \delta_{0,2}$, the system has a symmetry and motion is restricted to 2D invariant surfaces.

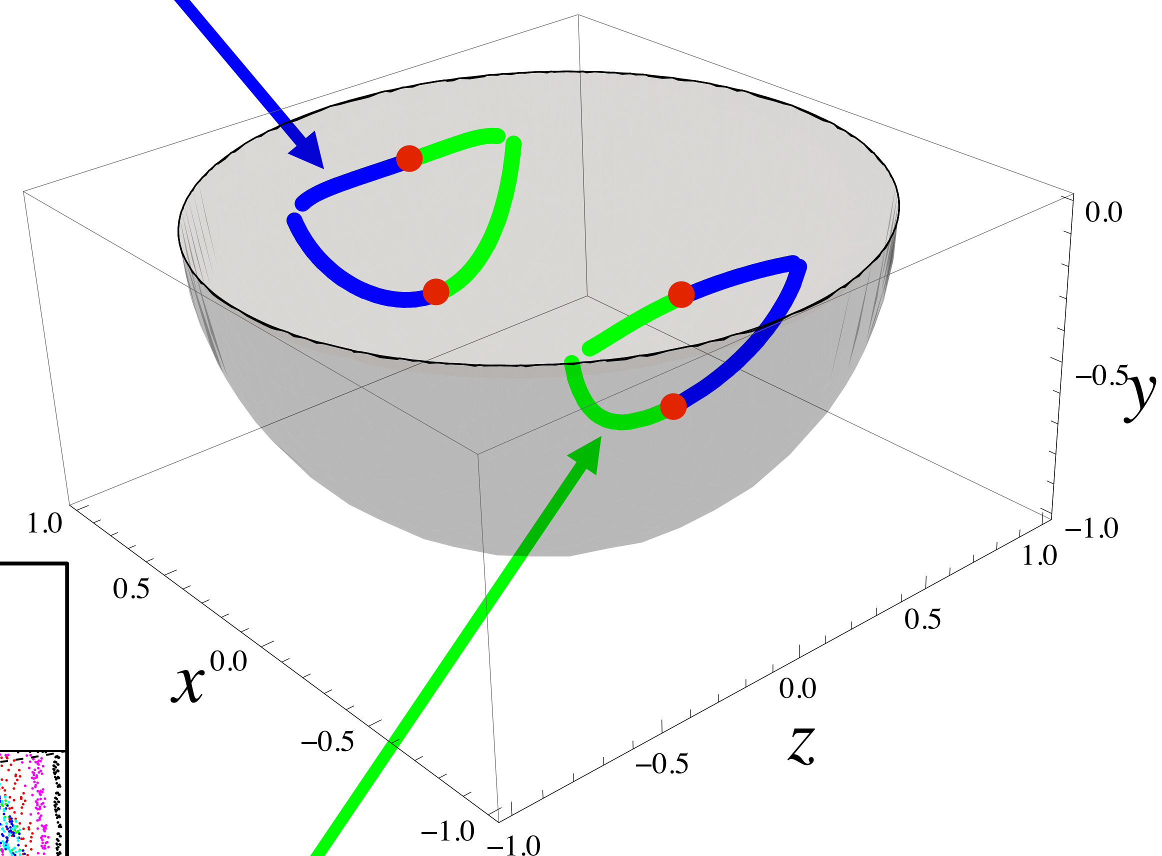
Period-1 points of the two-axis protocol

normally-elliptic curves of period-1 points





half-full quasi-2D



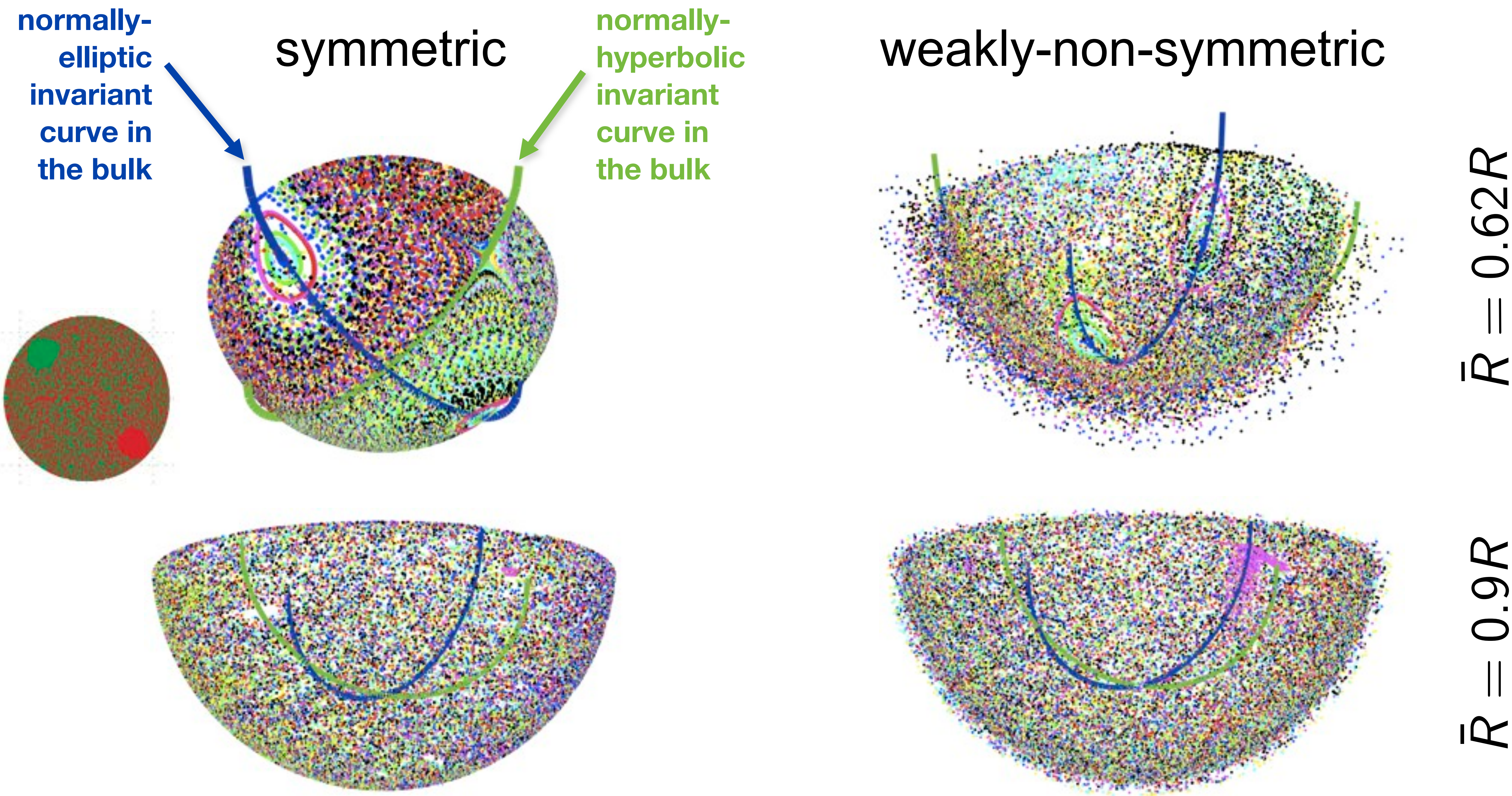
$\delta_{0,1} = \delta_{0,2}, \qquad \theta_1 = \theta_2$

$\delta_{0,1} \neq \delta_{0,2}, \qquad \theta_1 \neq \theta_2$

normally-hyperbolic curves of period-1 points

☞ Curves change stability type at a parabolic point.

Poincaré sections on the invariant surfaces and in-between

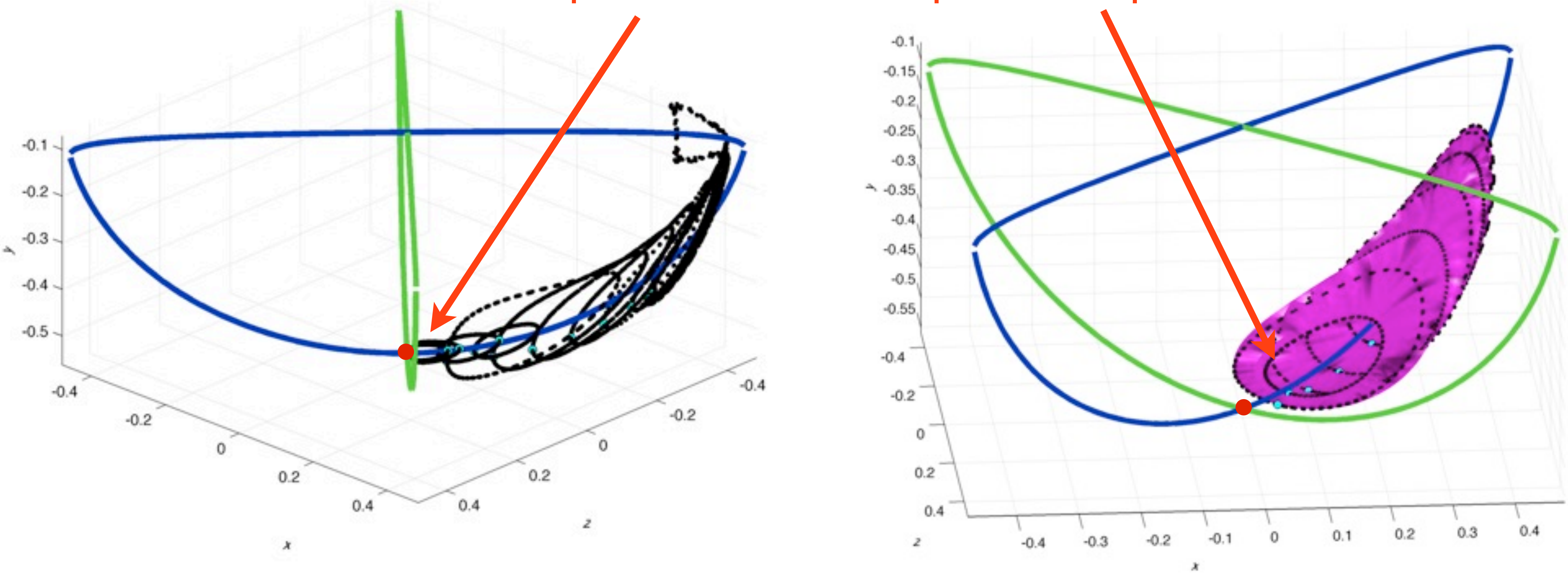


☞ Similar to effects of inertia in viscous laminar flows (Speetjens, Clercx, van Heijst, *et al.*).

"KAM-like" tubes: 3D barriers to transport

Idea: In the symmetric case, construct a KAM-like tube by stacking the "islands" from adjacent invariant surfaces.

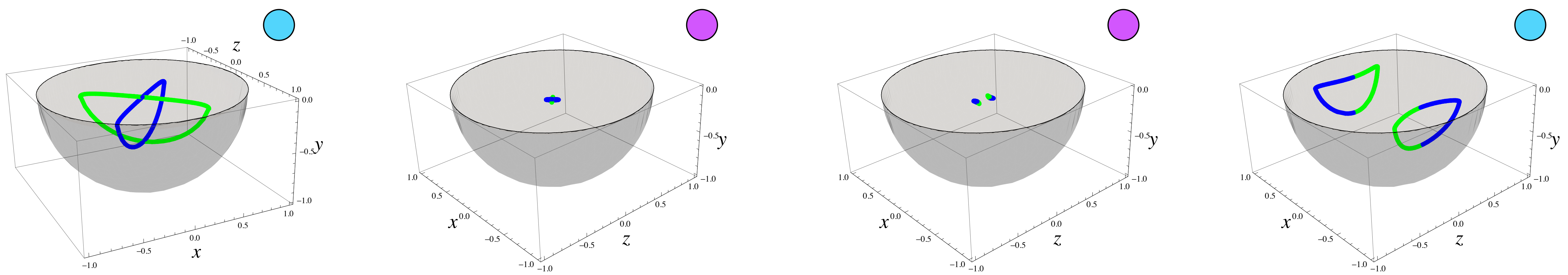
tube pinches-off at the parabolic point



(a) view 1 (b) view 2

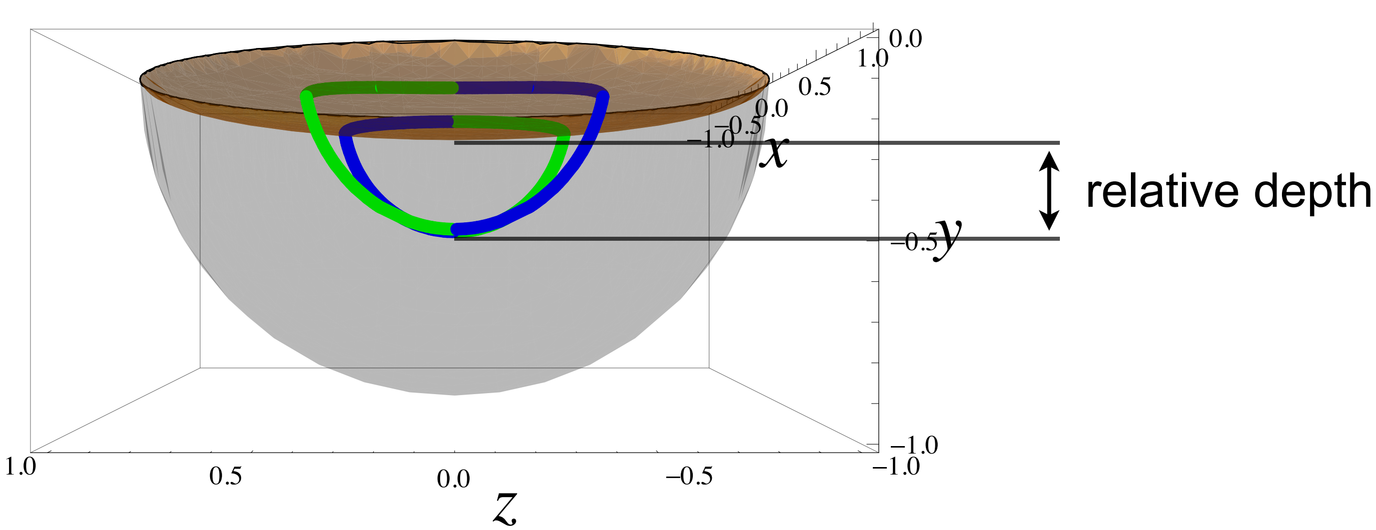
When do period-1 structures matter?

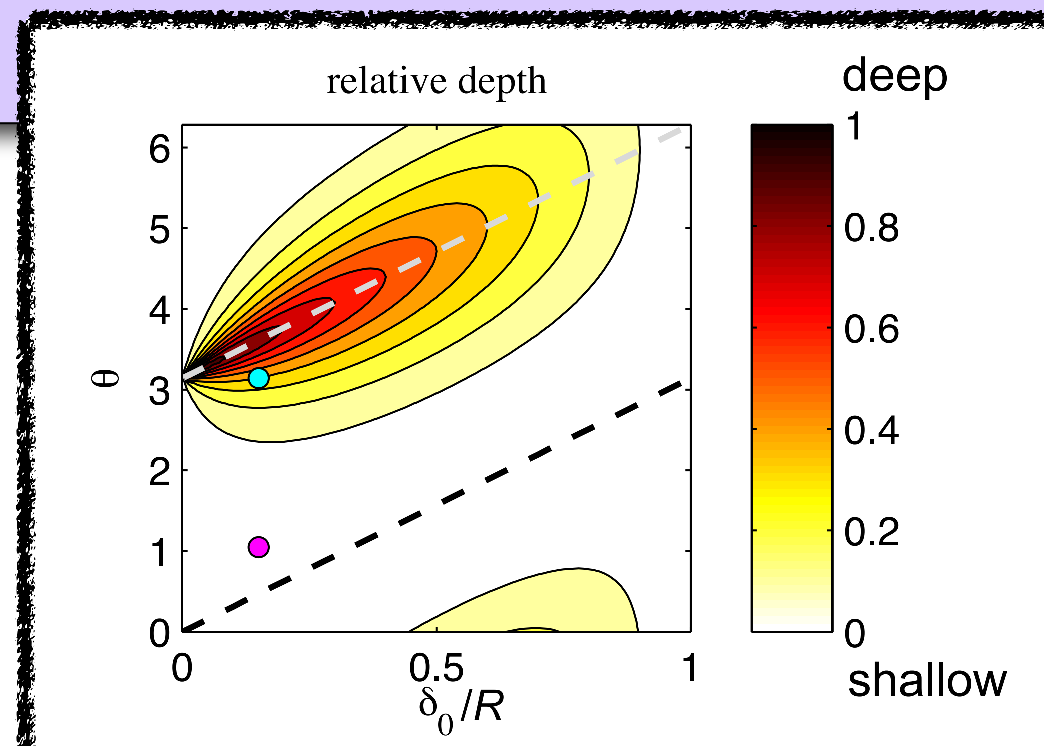
$(\delta_0 = 0.15)$



$\theta_1 = \theta_2 = \pi$ | $\theta_1 = \theta_2 = \pi/3$ | $\theta_1 = \pi/3,\ \theta_2 = \pi/2$ | $\theta_1 = 5\pi/4,\ \theta_2 = \pi$

- **<u>Thm:</u>** P-1 invariant curves' depth is max. at $\theta_{max} = (1+\delta_0/R)\pi$ and min. at $\theta_{min} = (\delta_0/R)\pi$.

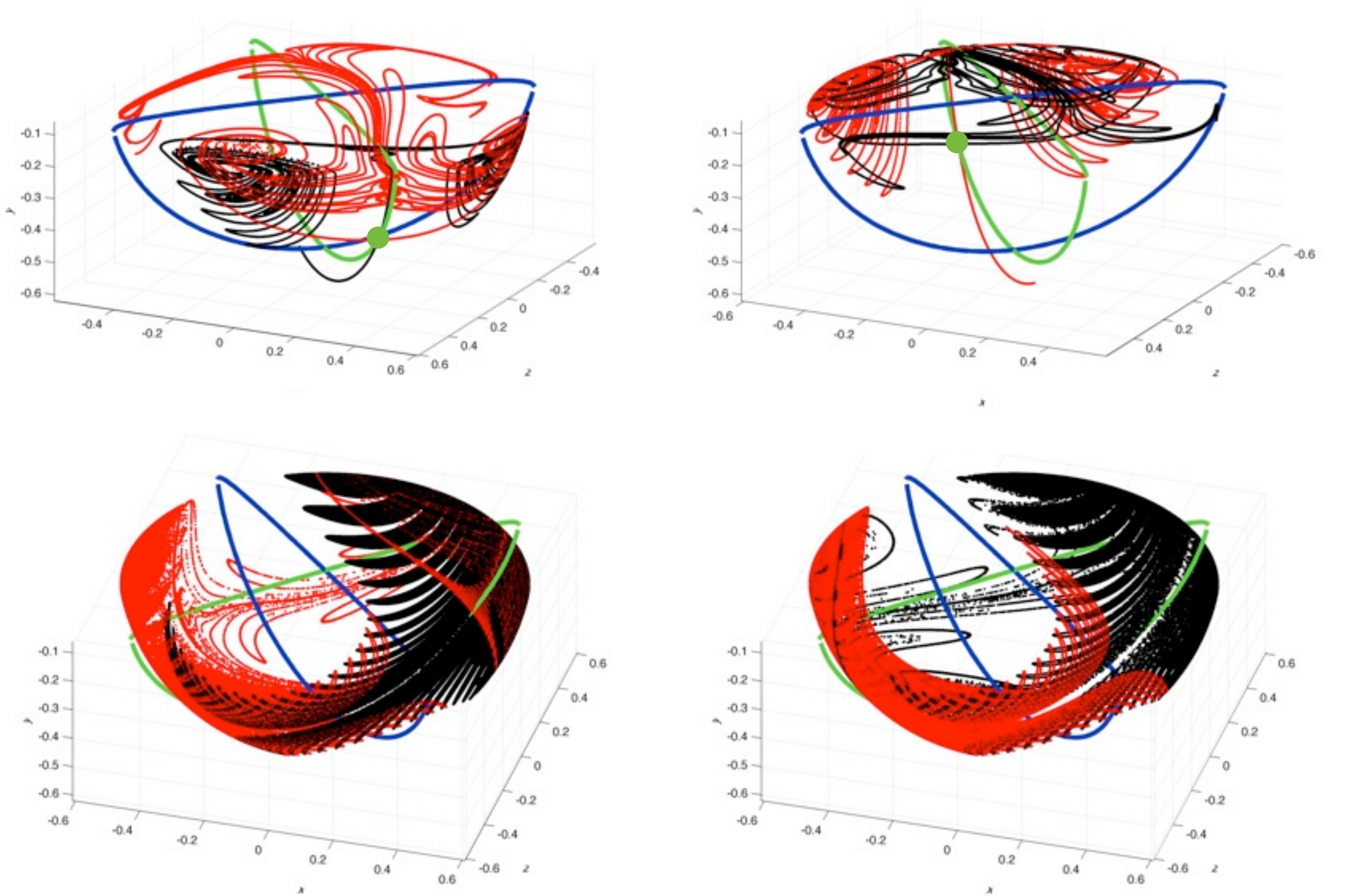




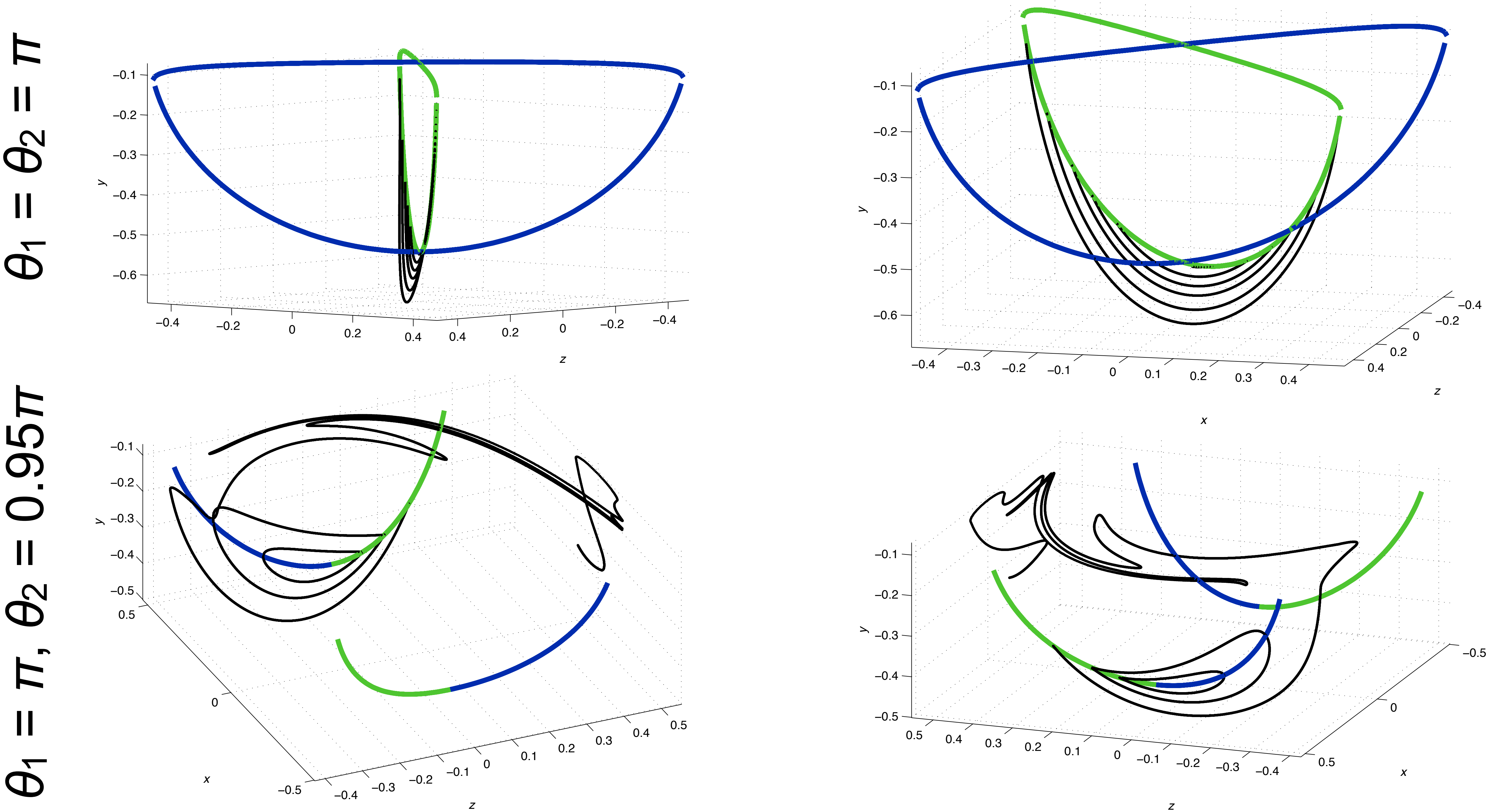
Manifolds: Progenitors of transport ($\omega_1 = \omega_2$, $\theta_1 = \theta_2$)

stable & unstable manifolds starting in the bulk

stable & unstable manifolds starting in the flowing layer



Taking a closer look: heteroclinic and homoclinic trajectories

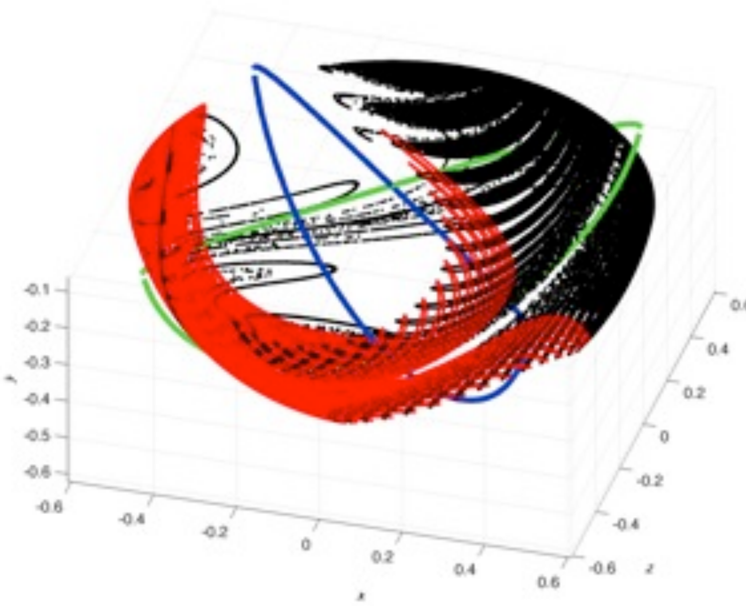


(a) view 1 (b) view 2

Summary & Future Work

- Dynamical systems theory can tell us a lot about granular mixing.
- In 3D, new possibilities emerge:
  - Curves of normally-elliptic and -hyperbolic points.
  - “KAM-like” tubes present barriers to transport.
  - Must break invariant surfaces to allow for fully-3D transport.
  - Manifolds structure is different from a perturbed Hamiltonian system.
- Challenges remain:
  - Visualization of 3D transport: “KAM-like” tubes in the flowing layer, 2D manifolds in the non-symmetric case, lobe dynamics...
  - Extension to non-circular geometries.

Thank you for your attention!



Questions?