# The Effect of Diffusion on Mixing by Cutting and Shuffling

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## Cutting and Shuffling of a Line Segment

- Much like the "mixing" of a deck cards, interval exchange transformations (IETs) represent cutting and shuffling [Krotter et al., 2012].
- In this case, "cards" can have nonuniform lengths and they may have a "color" assigned to them, both of which may change during the mixing process.
- Consider a cutting and shuffling IET construction with the following parameters:
  1. number of subsegments, $N$, introduced in each cutting step,
  2. shuffling order, which is a permutation, $\Pi$, of the integers up to $N$,
  3. lengths of each subsegment (parametrized by a fixed adjacent subsegment length ratio $r$),
  4. number of iterations performed, an integer $T$.

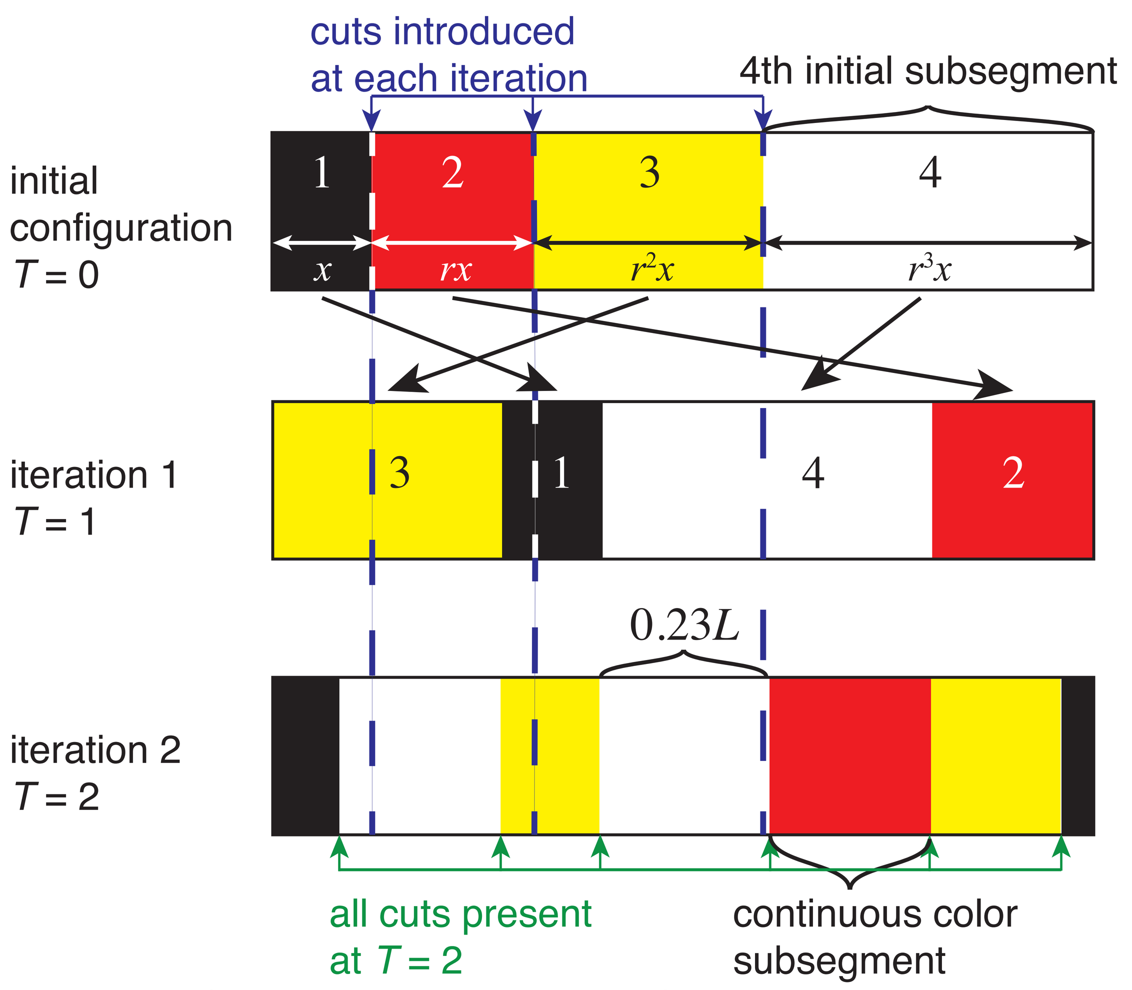


FIGURE 1: Example of a line segment composed of $N = 4$ subsegments with lengths $r^{i-1}x$, $i = 1, 2, 3, 4$. Two iterations of the cutting and shuffling process (without diffusion) are performed with the permutation $\Pi = [3142]$. Key terminology is labeled.

## Incorporating Diffusion into Cutting and Shuffling

- Consider a generic diffusion equation for the concentration $c(x, t)$ of subsegment color, with diffusivity $D$: $\partial_t c = D\partial_x^2 c$. Idea: Solve the diffusion eq. between c&s steps [Ashwin et al., 2002].
- Discretize the diffusion eq. with usual forward-time, central-space scheme on a lattice of unit spacings ($\Delta x = \Delta t = 1$) with periodic BCs: $c_i^{n+1} - c_i^n = D\big(c_{i+1}^n - 2c_i^n + c_{i-1}^n\big)$. Stable if $D \leq \frac{1}{2}$.
- This gives rise to a diffusion rule: $c_i \mapsto (1 - 2D)c_i + Dc_{i+1} + Dc_{i-1}$.
- Different $r \Rightarrow$ different $L$; use dimensional analysis to connect $T_{\text{final}}$ for different $L$ with fixed $D$.
- Finally, we use the standard $L^p$ norm, $||c||_p(T)$ ($p = 2$ for all examples here), to measures how far the segment's color distribution $c_j = c(x_j, t)$ is from the average (uniform) concentration $\bar{c}$:

$$||c||_p(T) = \left(\frac{\sum_{j=1}^{k(T)} |(c_j - \bar{c})|^p l_j}{\sum_{j=1}^{k(T)} l_j}\right)^{1/p},$$

where $k(T)$ is the number of continuous subsegments after $T$ iterations, each of length $l_j$, $c_j$ is the color of the $j$th continuous subsegments, $\bar{c}$ is the "average color" at $T = 0$.

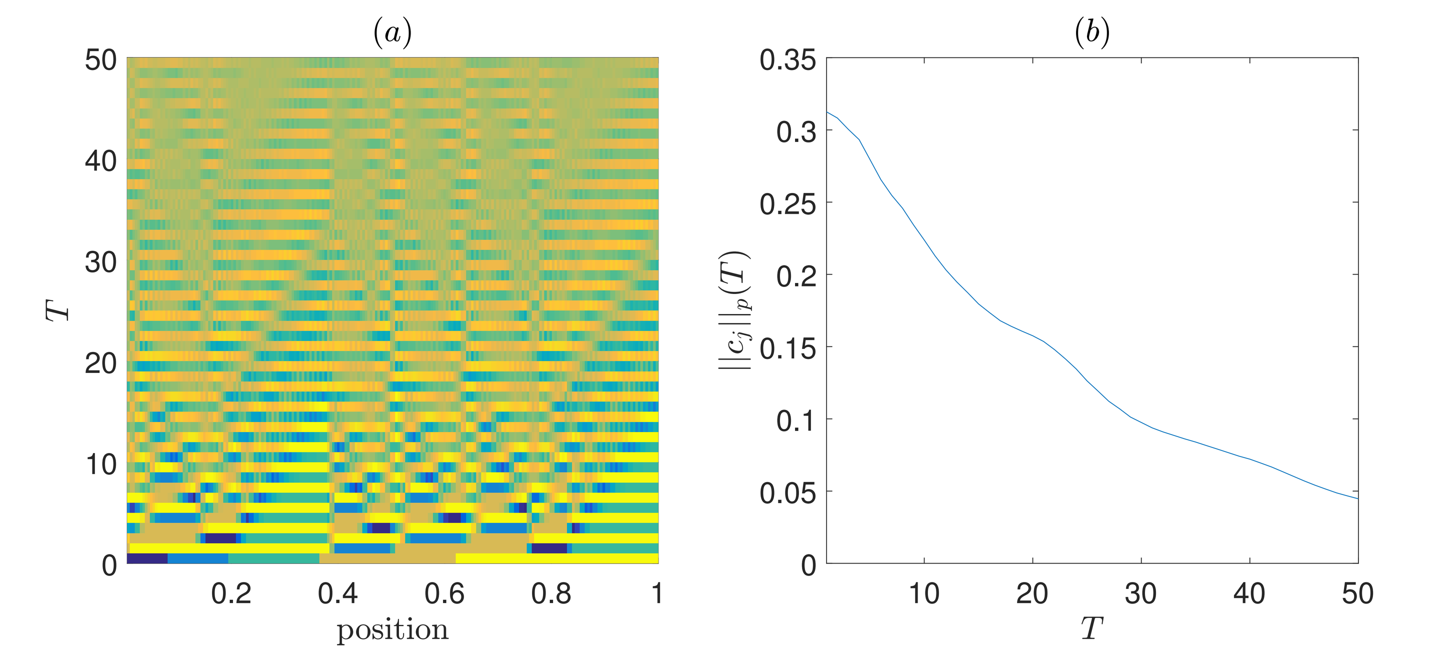


FIGURE 2: Example of the mixing process with $N = 5$, $\Pi = [52413]$, $r = 1.5$, $T = 50$, $D = 0.5$. (a) Space-time plot of mixing by cutting and shuffling with diffusion. (b) The mixing norm $||c||_p$ decays with $T$ due to diffusion.

## Rescaled Universal Mixing Curves and Cut-offs

- In card shuffling, [Aldous and Diaconis, 1986] used probabilistic methods to show 7 riffle shuffles are enough to randomize a deck; further shuffles do not produce significant mixing. This is a "cut-off."
- Question: Do cut-offs exist in IETs with diffusion? Preliminary work on chaotic mixing [Liang and West, 2008] suggests "probably." What about cutting and shuffling?
- Idea:
  1. Compute the mixing norm $||c_p||(T)$ for many different permutations $\Pi$, then average the curves into a single profile for each $N$ and $r$ (all with $D = 0.5$).
  2. Define $T_{Pe}$ as the number of iterations required for $||c_p||(T)$ to decay by a factor of $e^{-1}$. This value can be calculated exactly as the e-folding time $T_{Pe} = \tau\Gamma(1 + 1/\alpha)$, where $\alpha$ and $\tau$ are the fitting parameters of the mixing norm to the stretched exponential $M \cdot \exp\{(-T/\tau)^\alpha\}$.
     – Note: $Pe = DT_{\max}/L^2$ is a dimensionless diffusivity ("the Péclet number").
  3. Rescale $||c||_2(T) \mapsto ||c||_2(T)/M$, where $M := ||c||_2(0)$, so the mixing norm is in the range $[0, 1]$.
  4. Plot $||c||_2(T)/M$ vs. $T/T_{Pe}$, where $T_{Pe}$ is different for each curve. Now, we observe collapse onto a universal curve $\Rightarrow$ a possible cut-off!

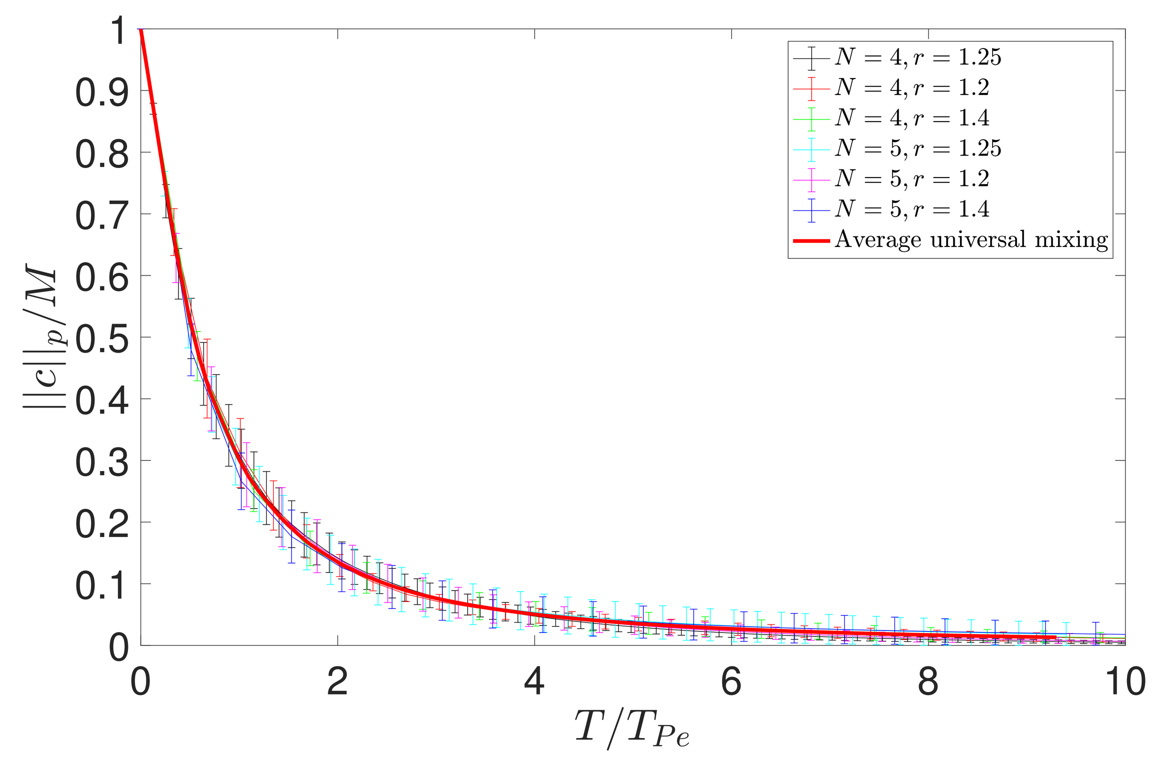


FIGURE 3: Cutting and shuffling with diffusion ($D = 0.5$) across different systems / mixing protocols exhibits a universal mixing behavior when rescaled, which suggests the existence of a "cut-off." Error bars in indicate a standard deviation from the mean.

## Predicting the "Stopping Time"

- Question: Can we predict the "stopping time" $T_{Pe}$ used to rescale $T$ in FIG. 3, if $Pe$ is given?
- Idea:
  1. From a simple 1D analysis of diffusion between two subsegments of unequal color color using an analytic solution to the diffusion equation [Schlick et al., 2013], we can estimate the subsegment length $\ell^*$ that would be "washed out" by diffusion in a characteristic time $\hat{T}$ as

$$\ell^* = \ell^*(\hat{T}) = \pi\sqrt{\frac{\hat{T}}{2Pe}}.$$

  2. From our cutting & shuffling simulations, we can calculate the *average* subsegment length $\ell_m$ without diffusion.
  3. We can require $\ell_m = \ell^*$ (the average subsegent will be "washed out" by diffusion in the same $T$ iterations) to get a solution $\tilde{T}_{Pe}$ (left panel in FIG. 4). This value is an estimate for stopping time $\tilde{T}_{Pe}$, given a $Pe$ value.
  4. We can now obtain the dependence of $\tilde{T}_{Pe}$ on $Pe$ (right panel in FIG. 4).

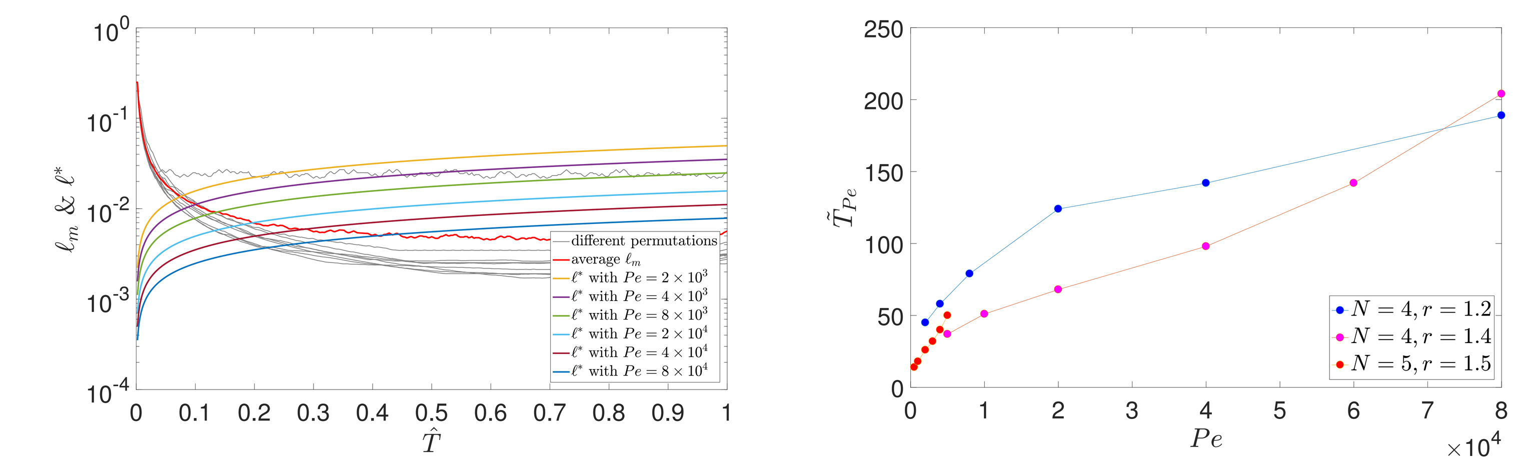


FIGURE 4: (Left) Computing the stopping time $\tilde{T}_{Pe}$ and (right) as a function of $Pe$.

⋆ For further information see our paper [Wang and Christov, 2018].

⋆ Outlook: Is universality present when stretching & folding is superimposed [Kreczak et al., 2017]? What about "optimizing" the cutting and shuffling process [Smith et al., 2018]?

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