# Cutting and Shuffling with Diffusion: Cut-offs in Interval Exchange Maps

Mengying Wang and Ivan C. Christov

School of Mechanical Engineering, Purdue University, West Lafayette, Indiana 47907, USA

## Cutting and Shuffling of a Line Segment

- Much like the "mixing" of a deck cards, interval exchange transformations (IETs) represent cutting and shuffling [Krotter et al., 2012].
- In this case, "cards" can have nonuniform lengths and they may have a "color" assigned to them, both of which may change during the mixing process.
- Cutting and shuffling is to be distinguished from stretching and folding (chaotic mixing) [Christov et al., 2011], and can arise when a granular material is broken into pieces and then put back together in a volume/area preserving (isometric) way [Juarez et al., 2010].
- Consider a cutting and shuffling IET construction with the following parameters:
  1. number of subsegments, $N$, introduced in each cutting step,
  2. shuffling order, which is a permutation, $\Pi$, of the integers up to $N$,
  3. lengths of each subsegment (parametrized by a fixed adjacent subsegment length ratio $r$),
  4. number of iterations performed, an integer $T$.

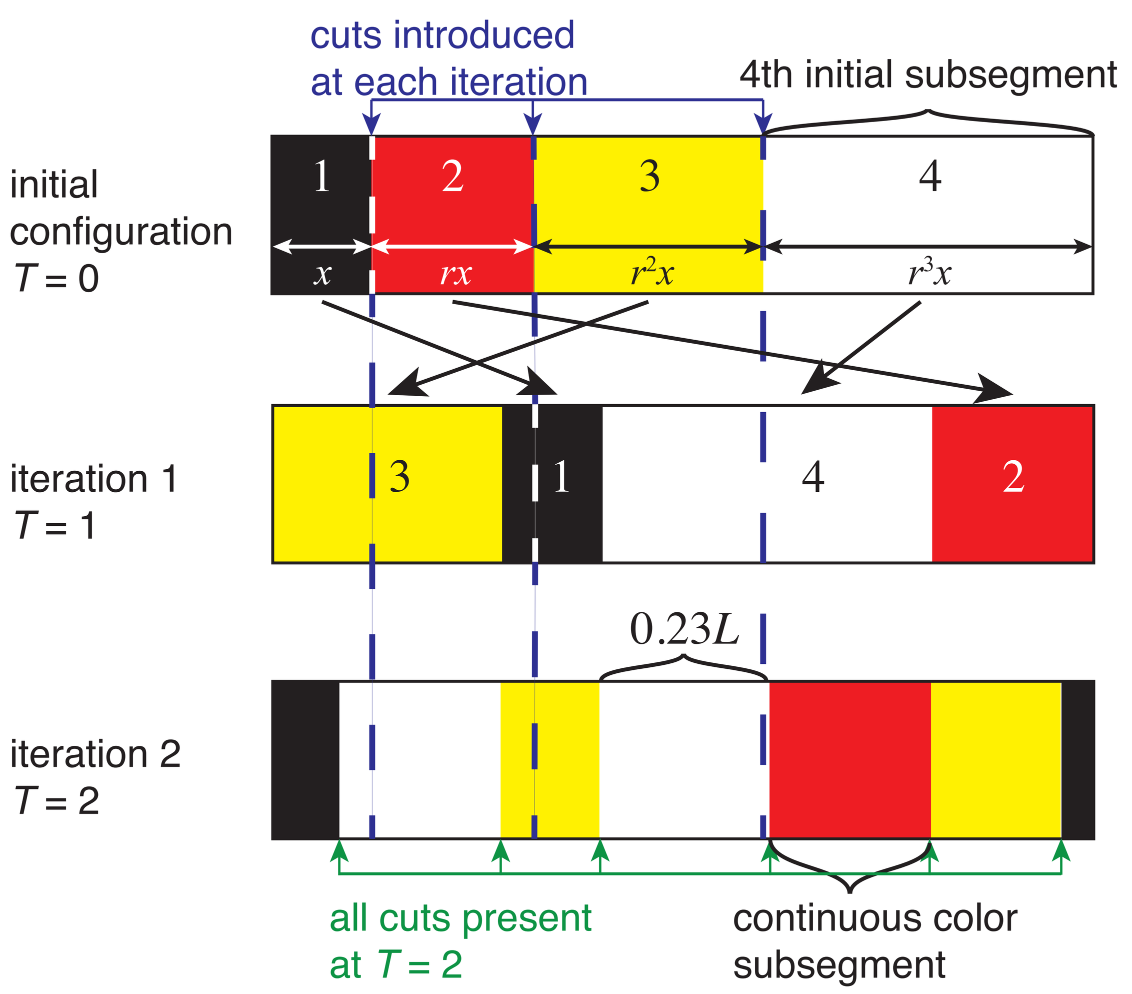


FIGURE 1: Example of a line segment composed of $N = 4$ subsegments with lengths $r^{i-1}x$, $i = 1, 2, 3, 4$. Two iterations of the cutting and shuffling process (without diffusion) are performed with the permutation $\Pi = [3142]$. Key terminology is labeled.

## Quantifying the Degree of Mixing

- Functional-space norms, including multiscale ones, are used to quantify mixing [Thiffeault, 2012].
- We use the standard $L^p$ norm, $||c_j||_p(T)$ ($p = 2$ for all examples here), to measures how far the segment's color distribution is from the average (uniform) concentration:

$$||c_j||_p(T) = \left( \frac{\sum_{j=1}^{k(T)} |(c_j - \bar{c})|^p l_j}{\sum_{j=1}^{k(T)} l_j} \right)^{1/p},$$

where $k(T)$ is the number of continuous subsegments after $T$ iterations, each of length $l_j$, $c_j$ is the color of the $j$th continuous subsegments, $\bar{c}$ is the "average color" at $T = 0$.

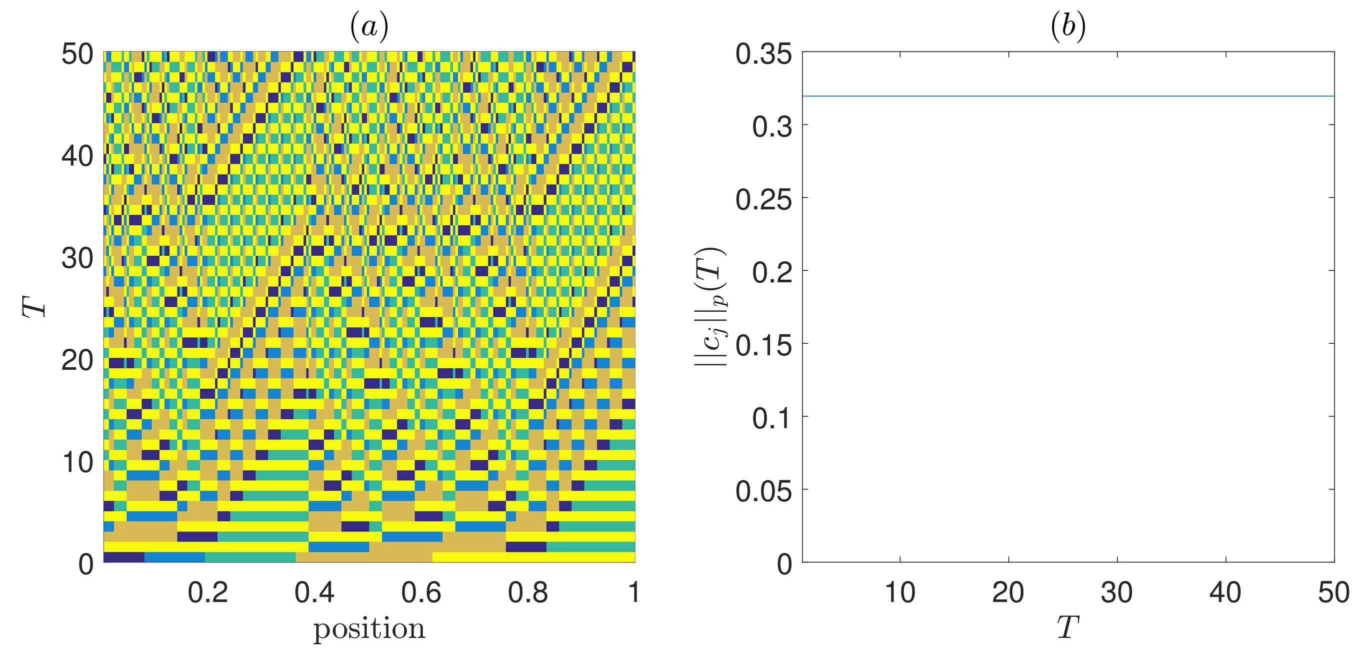


FIGURE 2: Cutting and shuffling process with $N = 5$, $\Pi = [52413]$, $r = 1.5$, $T = 50$. (a) Space-time plot of the mixing process. (b) The mixing norm remains constant without diffusion.

## Incorporating Diffusion

- Consider a generic diffusion equation for the concentration $c(x, t)$ of subsegment color, with diffusivity $D$: $\partial_t c = D\partial_x^2 c$. Idea: Solve the diffusion eq. between c&s steps [Ashwin et al., 2002].
- Discretize the diffusion eq. with usual forward-time, central-space scheme on a lattice of unit spacings ($\Delta x = \Delta t = 1$) with periodic BCs: $c_i^{n+1} - c_i^n = D(c_{i+1}^n - 2c_i^n + c_{i-1}^n)$, where $c_i^n \approx c(x_i, t^n)$ with $x_i = i$ ($i = 1, \dots, L$) and $t^n = n$ ($n = 0, \dots, T$). Stable if $D \leq \frac{1}{2}$.
- This gives rise to our diffusion rule: $c_i \mapsto (1 - 2D)c_i + Dc_{i+1} + Dc_{i-1}$.
- Different $r \Rightarrow$ different $L$; use dimensional analysis to connect $T_{\text{final}}$ for different $L$ with fixed $D$.

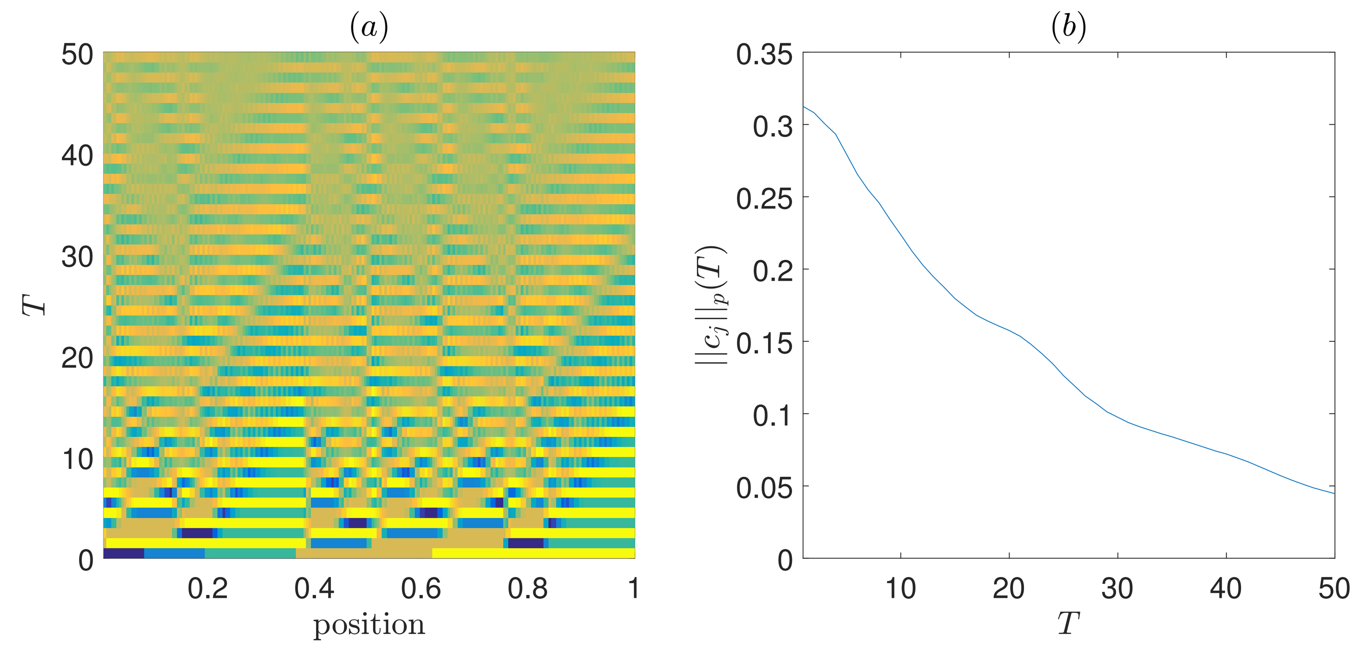


FIGURE 3: Example of incorporating diffusion into the same dynamical system from FIG. 2 with $D = 0.5$. (a) Space-time plot. (b) The mixing norm now decays with $T$ due to diffusion.

## Rescaled Universal Mixing Curves and Cut-offs

- In card shuffling, [Aldous and Diaconis, 1986] used probabilistic methods to show 7 riffle shuffles are enough to randomize a deck; further shuffles do not produce significant mixing. This is a "cut-off."
- Question: Do cut-offs exist in IETs with diffusion? Preliminary work on chaotic mixing [Liang and West, 2008] suggests "probably." What about cutting and shuffling?

- First, we compute the mixing norm, as a function of $T$, for many different permutations $\Pi$, then we average the curves into a single profile for each $N$ and $r$ (all with $D = 0.5$). Error bars in FIG. 4 show a standard deviation from the mean.
- Second, we find how many iterations (a number denoted by $T_{Pe}$) are required to decrease the initial value of the mixing norm by 50%: i.e., $||c_j||_2(T_{Pe}) \approx 0.5||c_j||_2(0)$.
- Third, we rescale $||c_j||_2(T) \mapsto ||c_j||_2(T)/M$, where $M := ||c_j||_2(0)$, so that the mixing norm is in the range $[0, 1]$.
- Fourth, we plot $||c_j||_2(T)/M$ vs. $T/T_{Pe}$, where $T_{Pe}$ is different for each curve, and we observe collapse onto a universal curve $\Rightarrow$ a cut-off!

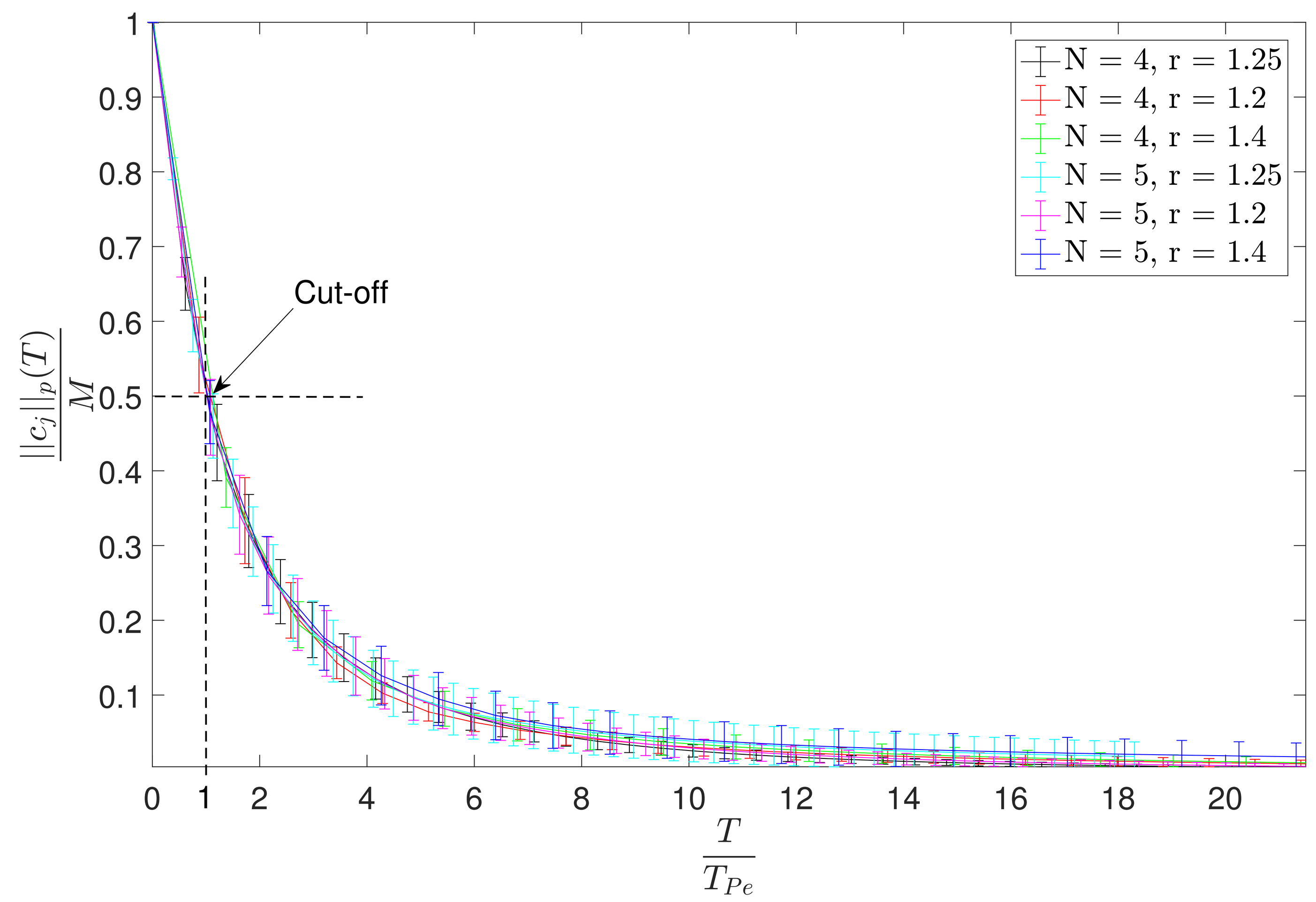


FIGURE 4: Cutting and shuffling with diffusion across different systems / mixing protocols exhibits a universal mixing behavior when rescaled, which suggests the existence of a "cut-off."

- Outlook: Is universality present when stretching & folding is superimposed [Kreczak et al., 2017]? What if diffusion is "optimized" [Froyland et al., 2016]?

## References

[Aldous and Diaconis, 1986] Aldous, D. and Diaconis, P. (1986). Shuffling cards and stopping times. *Am. Math. Monthly*, 93:333–348.

[Ashwin et al., 2002] Ashwin, P., Nicol, M., and Kirkby, N. (2002). Acceleration of one-dimensional mixing by discontinuous mappings. *Physica A*, 310:347–363.

[Christov et al., 2011] Christov, I. C., Lueptow, R. M., and Ottino, J. M. (2011). Stretching and folding versus cutting and shuffling: An illustrated perspective on mixing ... *Am. J. Phys.*, 74:359–367.

[Froyland et al., 2016] Froyland, G., González-Tokman, C., and Watson, T. M. (2016). Optimal mixing enhancement by local perturbation. *SIAM Rev.*, 58:494–513.

[Juarez et al., 2010] Juarez, G., Lueptow, R. M., Ottino, J. M., Sturman, R., and Wiggins, S. (2010). Mixing by cutting and shuffling. *EPL*, 91:20003.

[Kreczak et al., 2017] Kreczak, H., Sturman, R., and Wilson, M. C. T. (2017). Deceleration of one-dimensional mixing by discontinuous mappings. *Phys. Rev. E.*, 96:053112.

[Krotter et al., 2012] Krotter, M. K., Christov, I. C., Ottino, J. M., and Lueptow, R. M. (2012). Cutting and shuffling a line segment: Mixing by ... *Int. J. Bifurcation Chaos*, 22:1230041.

[Liang and West, 2008] Liang, T.-C. and West, M. (2008). Numerical evidence for cutoffs in chaotic microfluidic mixing. In *Proc. ASME 2008 Dynamic Systems and Control Conf.*, Ann Arbor, Michigan.

[Thiffeault, 2012] Thiffeault, J.-L. (2012). Using multiscale norms to quantify mixing and transport. *Nonlinearity*, 25:R1–R44.

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