Pattern formation and destruction in chaotic granular flows: Strange eigenmodes at finite Péclet number

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Eulerian Description of Mixing

- How does a passive tracer field (e.g., concentration) evolve at a given location in space?
- For a diffusive scalar $c(\vec{x},t)$ advected by a flow $\vec{v}$:

$$\partial_t c + \vec{v}(\vec{x},t)\cdot\nabla c = \mathrm{Pe}^{-1}\nabla^2 c, \qquad \text{Péclet number} = \mathrm{Pe} = VL/D.$$

- Advection-diffusion operator $\mathcal{L} = -\vec{v}\cdot\nabla + \mathrm{Pe}^{-1}\nabla^2$ is linear but not self-adjoint.
- Good news: $\mathcal{L}$ has discrete eigenvalues (point spectrum), i.e., $\mathcal{L}\hat{\phi}_k = \lambda_k\hat{\phi}_k$.
- Represent c by superposition $\approx \sum_k \alpha_k\hat{\phi}_k(\vec{x},t)e^{\lambda_k t}$, but eigenfunctions depend explicitly on t.
- All eigenvalues are such that $\Re\{\lambda_k\} < 0$, but those with $|\Re\{\lambda_k\}| \ll 1$ persist for long times $\Rightarrow$ strange eigenmodes [Pierrehumbert, 1994, Liu and Haller, 2004].
- For granular matter, need to model velocity field (see right column).

Floquet Theory for Advection-Diffusion (in a Nutshell)

- For a time-periodic $\vec{v}$, there exists a Floquet operator $\mathcal{F}$ [Liu and Haller, 2004] such that

$$\partial_t c = \mathcal{L}c \quad\Leftrightarrow\quad c(\vec{x},(n+1)T) = \mathcal{F}\{c(\vec{x},nT)\}.$$

- Using $\mathcal{F}\hat{\psi}_k = \mu_k\hat{\psi}_k$, eliminate the explicit time-dependance:

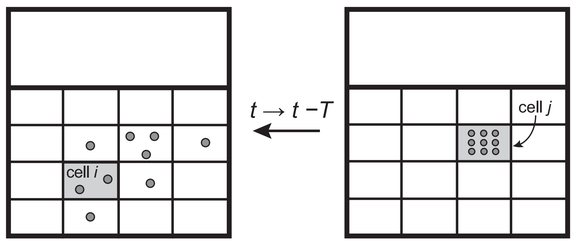
$$c(\vec{x},t) \approx \sum_k \alpha_k\hat{\varphi}_k(\vec{x},t)e^{\lambda_k t} \quad\Rightarrow\quad c(\vec{x},nT) = \sum_k \beta_k\hat{\psi}_k(\vec{x})\mu_k^n.$$

- Eigenvalues restricted to unit disk: $|\mu_k| < 1$. Eigenmodes with $|\mu_k| \approx 1$ persist for many periods.
- Can study structure of strange eigenmodes using a numerical analogue of $\mathcal{F}$ [Cerbelli et al., 2006].

Computational Approach to $\mathcal{F}$ and its Eigenstructure

- For $\mathrm{Pe}\to\infty$, $\mathcal{F}$ can be approximated on a grid by a matrix $\mathbf{\Phi}$ such that

$$\Phi_{ij} = \frac{\text{number of tracers in cell } j \text{ that came from cell } i}{\text{total number of tracers seeded in cell } j}.$$



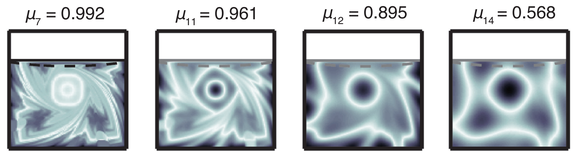
(a) tracers one period earlier (b) current location of tracers

- This is known as the mapping method for scalar transport (pure advection) [Singh et al., 2009].
- We incorporate "leakiness" between cells due to diffusion ($\mathrm{Pe} < \infty$) by analogy to operator splitting:

$$\frac{c^* - c^n}{T} = (-\vec{v}\cdot\nabla)c^*, \qquad \frac{c^{n+1} - c^*}{T} = (\mathrm{Pe}^{-1}\nabla^2)c^{n+1}$$

$$\Rightarrow\quad c^{n+1} = \left(\mathbf{I} - T\mathrm{Pe}^{-1}\nabla^2\right)^{-1}\underbrace{\mathbf{\Phi}c^n}_{c^*} \equiv \widetilde{\mathbf{\Phi}}c^n.$$

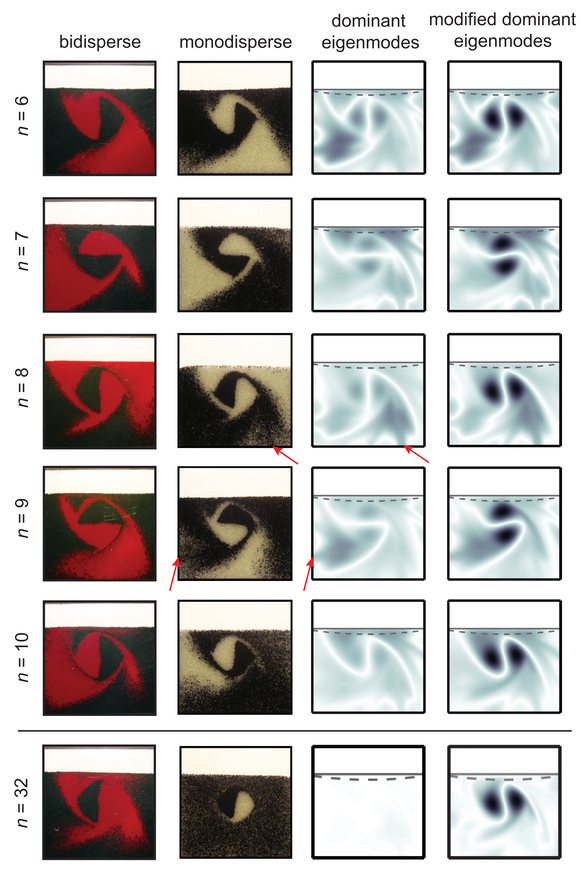
- Eigenvalues and eigenvectors of the matrix $\widetilde{\mathbf{\Phi}}$ approximate μ_k and $\hat{\psi}_k$, i.e, those of $\mathcal{F}$ at finite Pe.
- Example: diffusion blurs the fine-scale structure of a representative strange eigenmode.



(a) $\mathrm{Pe}=\infty$ (b) $\mathrm{Pe}=10^5$ (c) $\mathrm{Pe}=10^4$ (d) $\mathrm{Pe}=10^3$

- Spatial organization can be related to Lagrangian structure (KAM islands in the Poincaré section, invariant manifolds) of the dynamical system $\mathrm{d}\vec{x}/\mathrm{d}t = \vec{v}(\vec{x},t)$.

Comparison: Eigenmodes & Granular Mixing Experiments



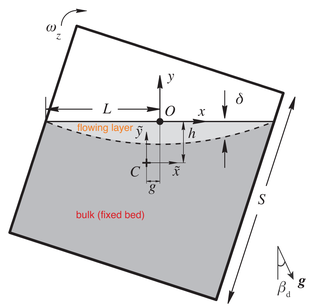
- Eigenmodes of the advection-diffusion Floquet operator $\mathcal{F}$ are excited in granular mixing.
- Our modified mapping method computational approach captures these patterns.
- Unlike fluid mixing [Rothstein et al., 1999], diffusion "destroys" strange eigenmodes within $\approx$ 2 dozen periods, for monodisperse granular materials.
- Segregation pattern of bidisperse granular materials takes $\approx$ the shape of the dominant eigenmodes.

Kinematics of Granular Flow in a Rotating Tumbler

- In the moving frame, simple shear in the flowing layer + solid body rotation in the bulk leads to

$$v_x(x,y,t) = \begin{cases} 2\bar{v}_x(t)\left[1 + y/\delta(x,t)\right], & y > -\delta(x,t); \\ \omega_z[y + h(t)] - \dot{g}(t), & \text{otherwise.} \end{cases}$$

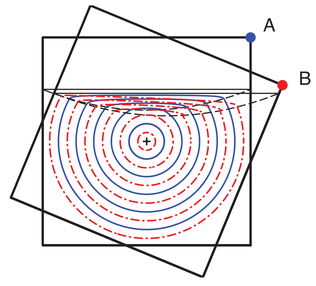
$$v_y(x,y,t) = \begin{cases} -\omega_z x[y/\delta(x,t)]^2, & y > -\delta(x,t); \\ -\omega_z[x + g(t)] - \dot{h}(t), & \text{otherwise.} \end{cases}$$



- Depth-averaged streamwise velocity $\bar{v}_x \neq \bar{v}_x(x)$ & conservation of mass [Khakhar et al., 1999] give

$$\delta(x,t) = \delta_0(t)\left\{1 - [x/L(t)]^2\right\},$$
$$\bar{v}_x(t) = \omega_z L(t)^2/[2\delta_0(t)].$$

- Time-periodic flow: symmetries of the container lead to periodicity: $v_{x,y}(x,y,t) = v_{x,y}(x,y,t+T)$, where $T = (2\pi/\omega)/(\mathrm{n\ sides})$ is the flow period.
- Streamline crossing criterion: to achieve chaotic mixing, superimposed streamlines at two different times should intersect.
- Changing length and depth of flowing layer $\Rightarrow$ streamline crossing in a quasi-2D square tumbler.
- $\delta_0 \sim$ 5 to 12 particle diameters.



Acknowledgements: Thanks to Emre Yildiz and Florent Pignatel for experimental help. Research supported, in part, by funding from NSF Grant CMMI-1000469. Results have been published in *Physics of Fluids* [Christov et al., 2011].

References

[Cerbelli et al., 2006] Cerbelli, S., Adrover, A., Creta, F., and Giona, M. (2006). Foundations of laminar chaotic mixing and spectral theory of linear operators. *Chem. Eng. Sci.*, 61:2754–2761.

[Christov et al., 2011] Christov, I. C., Ottino, J. M., and Lueptow, R. M. (2011). From streamline jumping to strange eigenmodes: Bridging the lagrangian and eulerian pictures of the kinematics of mixing in granular flows. *Phys. Fluids*, 23:103302.

[Khakhar et al., 1999] Khakhar, D. V., McCarthy, J. J., Gilchrist, J. F., and Ottino, J. M. (1999). Chaotic mixing of granular materials in two-dimensional tumbling mixers. *Chaos*, 9:195–205.

[Liu and Haller, 2004] Liu, W. and Haller, G. (2004). Strange eigenmodes and decay of variance in the mixing of diffusive tracers. *Physica D*, 188:1–39.

[Pierrehumbert, 1994] Pierrehumbert, R. T. (1994). Tracer microstructure in the large-eddy dominated regime. *Chaos Solitons Fractals*, 4:1091–1110.

[Rothstein et al., 1999] Rothstein, D., Henry, E., and Gollub, J. P. (1999). Persistent patterns in transient chaotic fluid mixing. *Nature*, 401:770–772.

[Singh et al., 2009] Singh, M. K., Speetjens, M. F. M., and Anderson, P. D. (2009). Eigenmode analysis of scalar transport in distributive mixing. *Phys. Fluids*, 21:093601.

○ Presented at the Poster Sessions of Dynamics Days 2012, Baltimore, Maryland. ○