Chaotic granular mixing in quasi-two-dimensional tumblers: Streamline jumping and piecewise isometries

Ivan C. Christov[1], Julio M. Ottino[2,3,4] and Richard M. Lueptow[4]

[1]Department of Engineering Sciences and Applied Mathematics, [2]Department of Chemical and Biological Engineering,
[3]The Northwestern Institute on Complex Systems (NICO), [4]Department of Mechanical Engineering,
Robert R. McCormick School of Engineering and Applied Science, Northwestern University, Evanston, IL

NORTHWESTERN UNIVERSITY

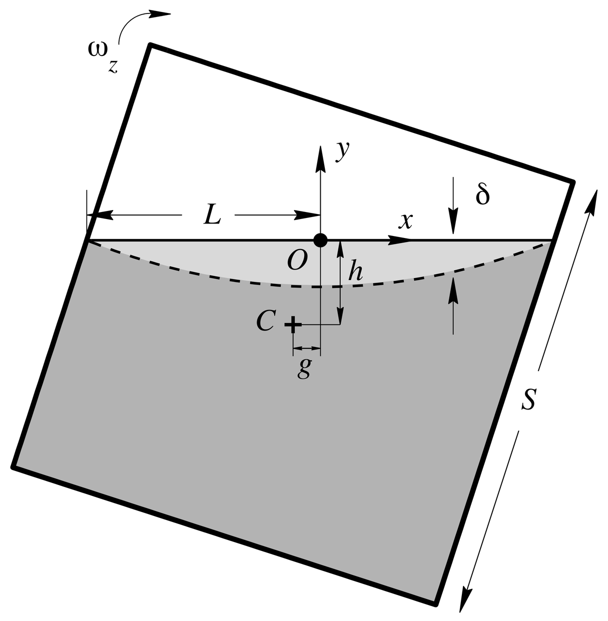
McCormick Northwestern Engineering

Dynamical Systems Framework for Tumbled Granular Flow

- Kinematic approach to mixing [Ott90] in a quasi-2D tumbled granular flow leads to the dynamical system

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \begin{cases} v_x(x(t), y(t), t), & y(t) > -\delta(x(t), t);\\ \omega_z[y(t) + h(t)] - \dot{g}(t), & \text{otherwise.}\end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = \begin{cases} v_y(x(t), y(t), t), & y(t) > -\delta(x(t), t);\\ -\omega_z[x(t) + g(t)] - \dot{h}(t), & \text{otherwise.}\end{cases}$$

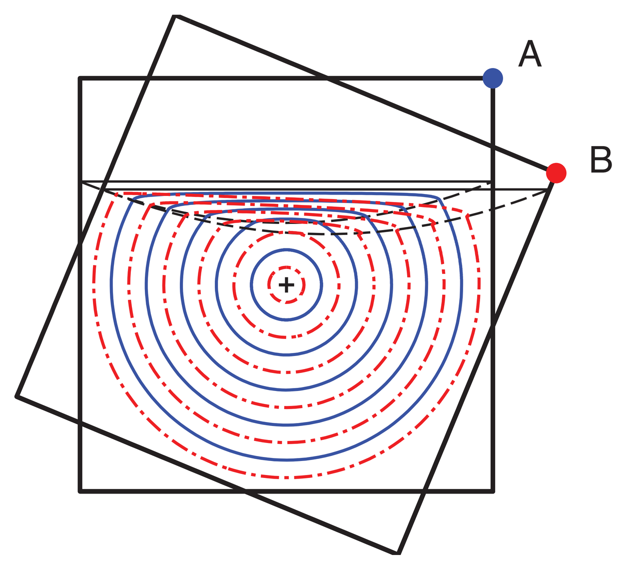


- Depth-averaged streamwise velocity $\bar{v}_x \neq \bar{v}_x(x)$ & conservation of mass [KMGO99] give

$$v_x(x, y, t) = 2\bar{v}_x(t)\,[1 + y/\delta(x, t)]\,, \qquad v_y(x, y, t) = -\omega_z x\,[y/\delta(x, t)]^2\,,$$

$$\delta(x, t) = \delta_0(t)\left\{1 - [x(t)/L(t)]^2\right\}, \qquad \bar{v}_x(t) = \omega_z L(t)^2/[2\delta_0(t)].$$

- Streamline crossing criterion: to achieve chaotic mixing, superimposed streamlines at two different times should intersect.
- Changing length and depth of flowing layer ⇒ streamline crossing in a quasi-2D square tumbler.
- Geometric similarity: $\epsilon := \delta_0(t)/L(t) = const.$ ⇒ flowing layer adjusts instantaneously to changes in container's orientation.
- $\delta_0 \sim 5$ to 12 particle diameters ⇒ $\epsilon \ll 1$.



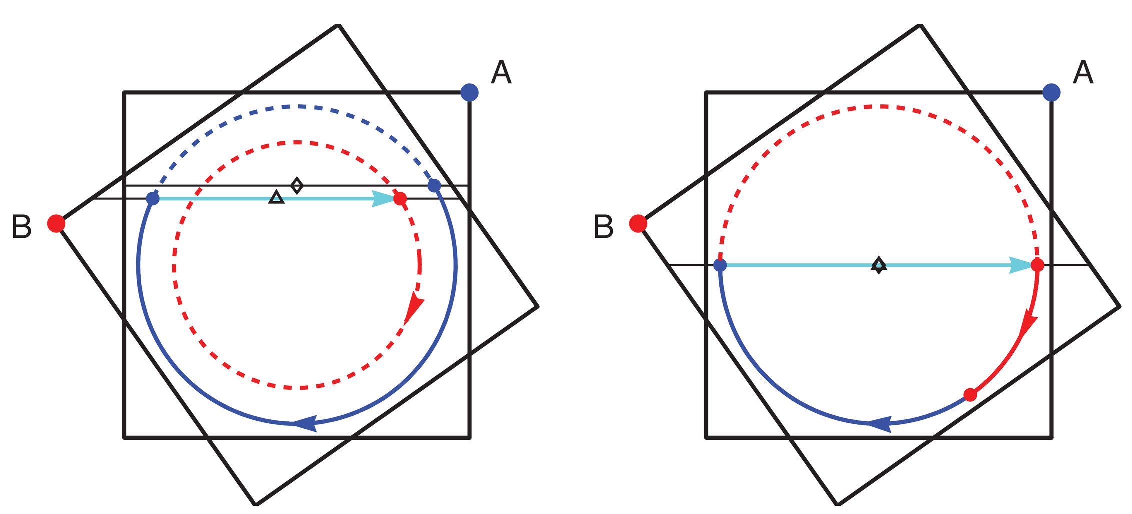
- If streamline crossing (⇒chaotic advection) were the only mixing mechanism, then as $\epsilon \to 0$ the flow should become trivial. However, this is not true [EV99]. **Why?**

Streamline Jumping: A New Mixing Mechanism

- $\lim_{\epsilon\to 0} \delta_0 = 0$ ⇒ flowing layer collapses onto the free surface becoming an interface.
- The average speed of a particle on the free surface becomes infinite:

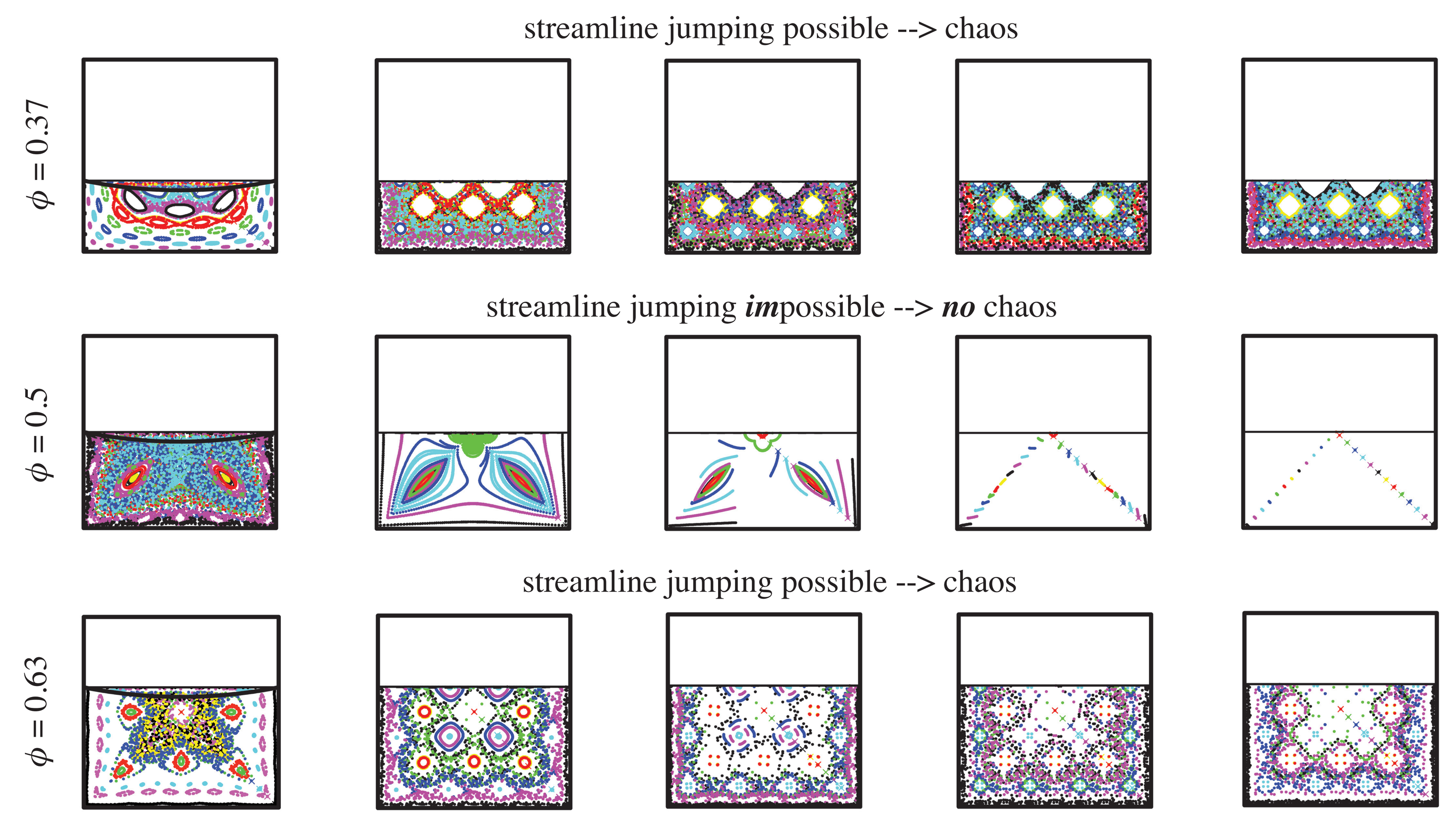
$$\bar{V}_{\text{surf}} = \frac{1}{T_f}\int_0^{T_f} \frac{1}{2L}\int_{-L}^{+L} \sqrt{v_x^2 + v_y^2}\,\bigg|_{y=0} \mathrm{d}x\,\mathrm{d}t = \frac{\bar{L}}{\epsilon} \to \infty \quad \text{as} \quad \epsilon \to 0.$$

- For consistency $x_{\text{exit}} = -x_{\text{enter}}$ when crossing the flowing layer, $\tilde{x}_{\text{exit}} = -\tilde{x}_{\text{enter}} + 2g(t_{\text{enter}})$ w.r.t. C.



(a) $\phi = 0.73$ (b) $\phi = 0.50$

- Important point: $g(t_{\text{enter}}) \neq 0$ ⇒ streamline jumping because $\tilde{x}_{\text{exit}} \neq -\tilde{x}_{\text{enter}}$!
- Streamline jumping is a necessary and sufficient condition for chaotic mixing in a convex tumbler, rotated at a constant speed, with a vanishingly-thin flowing layer.
- Can confirm this through 500-period Poincaré sections.



(a) $\varepsilon = 10^{-1}$ (b) $\varepsilon = 10^{-2}$ (c) $\varepsilon = 10^{-3}$ (d) $\varepsilon = 10^{-4}$ (e) $\varepsilon = 0$ Skeleton of the flow

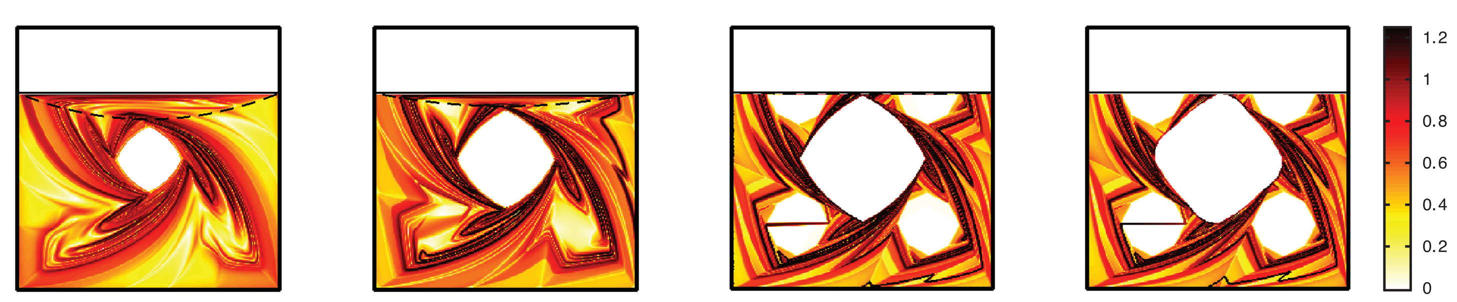
- Chaotic regions exist only for $\phi \neq 0.5$ as predicted.
- Important point: $\epsilon = 0$ Poincaré sections are the template or skeleton of nearby ones.

Finite-Time Lyapunov Exponents and Manifold Structure

- Poincaré sections reveal unmixed regions (KAM islands) but mixing is characterized by the unstable manifold [Ott90].
- So, consider the (largest) finite-time Lyapunov exponent (FTLE) field of the flow

$$\sigma(X, Y; t_0, \tau) = \frac{1}{|\tau|}\ln\sqrt{\Lambda_{\max}(\mathbf{C}(X, Y; t_0, \tau))},$$

- A Lagrangian coherent structure (LCS) is a ridge of σ [SLM05]; for a qualitatively analysis, can identify ridges with the darkest areas in the figure.



(a) $\varepsilon = 0.02$ (b) $\varepsilon = 0.01$ (c) $\varepsilon = 0.001$ (d) $\varepsilon = 0$ Skeleton of the flow

- Mass flux across an LCS is negligible [SLM05] ⇒ LCSs are finite-time analogues to the stable/unstable manifolds of the flows with arbitrary time-dependance.
- Important point: manifold structure of the $\epsilon = 0$ system is the template or skeleton for all $\epsilon \ll 1$ ⇒ streamline jumping is the predominant mixing mechanism here.

Limiting Dynamics as a Piecewise Isometry

- The limiting ($\epsilon = 0$) dynamical system is a piecewise isometry (PWI) — a discontinuous dynamical system studied only recently [Goe02].
- PWIs exhibit the usual nonlinear dynamics: periodic points, quasi-periodicity, fractal structure, global attractors and generally complex dynamics.
- But, PWIs do not possess the stretching and folding (Smale horseshoe) mechanism that leads to chaos [Goe02, SMOW08].
- The PWI is an affine transformation $\mathbf{\Omega}(t_1, t_2) : \mathcal{D}(t_1) \to \mathcal{R}(t_2)$ that can be written as

$$\mathbf{\Omega}(t_1, t_2) = \mathbf{T}(t_2)\circ\mathbf{R}\circ\mathbf{Q}(t_1, t_2) = \begin{pmatrix} -\cos(\omega_z\bar{t}) & -\sin(\omega_z\bar{t}) & 2g(t_2)\\ -\sin(\omega_z\bar{t}) & \cos(\omega_z\bar{t}) & 0\\ 0 & 0 & 1\end{pmatrix}.$$

- $\text{eig}[\mathbf{\Omega}(t_1, t_2)] = \{1, 1, -1\}\ \forall t_1, t_2$ ⇒ no stretching or compression, yet complex dynamics persist due to the discontinuity.
- Cutting and shuffling dynamics: $\mathbf{Q}$ "shuffles" by mapping each point in the flowing layer to a new location, $\mathbf{R}\circ\mathbf{T}$ "cuts" by reflecting & translating points along the flowing layer.

Summary and Open Questions

✓ The vanishing-flowing layer limit of quasi-2D tumbled granular flows is well defined.

✓ The $\epsilon = 0$ mixing mechanism is streamline jumping.

✓ LCSs show the manifold structure of this non-smooth flow.

✓ Limiting dynamics can be framed using the new mathematics of PWIs.

◦ In 3D, what happens when multiple-axes rotation protocols are combined with streamline jumping?

◦ Can we study the ergodicity of PWIs and prove mathematically their mixing properties?

◦ Can we predict granular mixing *a priori* based on container geometry and PWIs?

References

[EV99] T. Elperin and A. Vikhansky. Chaotic mixing of granular material in slowly rotating containers as a discrete mapping. *Chaos*, 9:910–915, 1999.

[Goe02] A. Goetz. Piecewise isometries — an emerging area of dynamical systems. In P. Grabner and W. Woess, editors, *Fractals in Graz 2001*, pp. 135–144. Birkhäuser, Basel, 2002.

[KMGO99] D. V. Khakhar, J. J. McCarthy, J. F. Gilchrist, and J. M. Ottino. Chaotic mixing of granular materials in two-dimensional tumbling mixers. *Chaos*, 9:195–205, 1999.

[Ott90] J. M. Ottino. Mixing, chaotic advection, and turbulence. *Annu. Rev. Fluid Mech.*, 22:207–253, 1990.

[SLM05] S. C. Shadden, F. Lekien, and J. E. Marsden. Definition and properties of Lagrangian coherent structures from finite-time Lyapunov exponents in two-dimensional aperiodic flows. *Physica D*, 212:271–304, 2005.

[SMOW08] R. Sturman, S. W. Meier, J. M. Ottino, and S. Wiggins. Linked twist map formalism in two and three dimensions applied to mixing in tumbled granular flows. *J. Fluid Mech.*, 602:129–174, 2008.

◦ Presented at the Poster Sessions of Dynamics Days 2010, Evanston, Illinois. ◦