Dynamics of Quasi-Particles in Nonlinear Wave Equations

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The sine-Gordon Equation and its Soliton Solutions

- The sine-Gordon equation was the first ever "model unified field equation" because, as shown by Perring & Skyrme [PS62], its solutions can be interpreted as interacting mesons and baryons. Its dimensionless form reads

$$u_{tt} - u_{xx} = -\sin u. \tag{1}$$

- It is the Euler–Lagrange equation of the field described by the Lagrangian and Hamiltonian

$$L = \frac{1}{2}\int_{-\infty}^{+\infty} u_t^2 - u_x^2 - 2(1-\cos u)\,\mathrm{d}x, \quad H = \frac{1}{2}\int_{-\infty}^{+\infty} u_t^2 + u_x^2 + 2(1-\cos u)\,\mathrm{d}x. \tag{2}$$

- Though Perring & Skyrme postulated that the solutions of (1) behaved like particles, the latter assumptions was vindicated by Zabusky & Kruskal's discovery of **solitons** and their particle-like interactions [ZK65]. In addition, the solutions given in [PS62] have since been shown to be, indeed, **two-soliton solutions** of the sine-Gordon equation [BC80]. The latter read

$$u^+ = 4\arctan\left\{\frac{\cosh[(\theta_1-\theta_2)/2]}{a_{12}\sinh[(\theta_1+\theta_2)/2]}\right\}, \qquad u^- = 4\arctan\left\{\frac{\sinh[(\theta_1-\theta_2)/2]}{a_{12}\cosh[(\theta_1+\theta_2)/2]}\right\}, \tag{3}$$

where, for $i \in \{1,2\}$,

$$\theta_i = \gamma_i\,(x - c_i t - x_i)\,, \quad \gamma_i^2 = \left(1-c_i^2\right)^{-1}, \qquad a_i^2 = (1-c_i)\,(1+c_i)^{-1}, \quad a_{12} = |a_1-a_2|^2|a_1+a_2|^{-2}. \tag{4}$$

It is customary to call u^+ and u^- the soliton-soliton (S-S) and soliton-antisoliton (S-A) solutions, respectively. Furthermore, it is well-known that, for well-separated solitons, (3) reduces to the **linear superpositon** of two **single** soliton solutions:

$$u^{\pm}(x,t) \equiv 4\arctan\big[\exp\big(\theta_1+\delta_1^{\pm}\big)\big] \pm 4\arctan\big[\exp\big(\theta_2+\delta_2^{\pm}\big)\big], \tag{5}$$

where $\delta_1^{\pm} = -\delta_2^{\pm} = \frac{1}{2}\ln a_{12}$ are the phase shifts of the individual solitons.

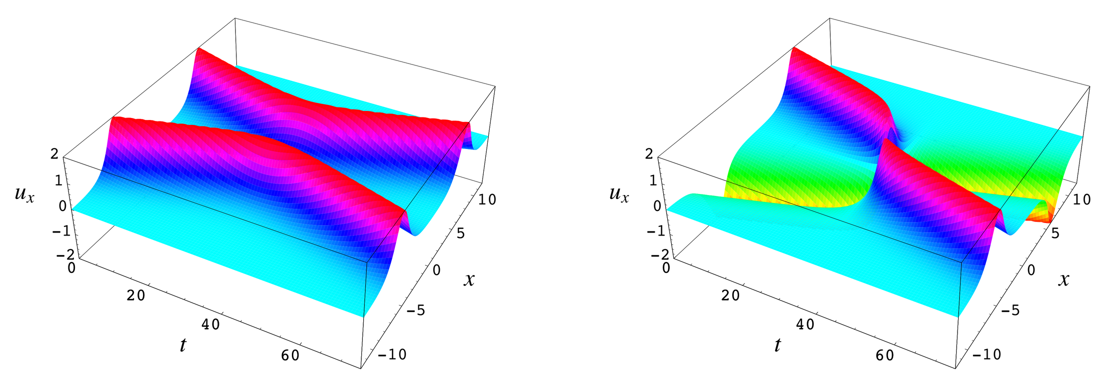

Figure 1: Space-time plots of the S-S (left) and S-A (right) solutions.

- Important terminology:
  - A solitary wave is a solution of a (nonlinear) wave equation that is localized in space (bump shape), or its derivative is localized in space (kink shape).
  - A quasi-particle is a point particle corresponding to a solitary wave that preserves or recovers its original shape after interaction or collision with another solitary wave.
  - A soliton is a solitary-wave solution of a fully-integrable nonlinear wave equation.
  - One can assign a quasi-particle to every type of soliton. However, not all quasi-particles correspond to solitons because non-integrable equations also have solitary-wave solutions that one can identify with quasi-particles.
- Thus, investigating the particle-like, discrete, dynamics of the solutions of physical models, beyond the well-known (integrable) "toy" models, is of extreme importance. In particular, in the case of non-integrable equations, approximate methods (such as the as the **coarse-grain description** proposed herein) are required to make any progress.

The Coarse-Grain Description

⋆ For the sake of simplicity, the rest of the discussion is restricted to the "classical" limit of the dynamics, *i.e.*, when the phase speeds c_i and accelerations $\dot{c}_i$ are $\ll 1$, and the shape of the solitary waves does not depend on their phase speeds.

- Idea: "Degrade" the full continuous description of the wave profile by treating the solitons as discrete structures. Mathematically speaking, we wish to write the solution as

$$u(x,t) = \Phi_1[x - X_1(t)] + \Phi_2[x - X_2(t)] + \Phi_{12}[x - X_1(t), x - X_2(t)], \tag{6}$$

where Φ_1 and Φ_2 are the (non-interacting) shapes of single solitons (quasi-particles) and Φ_{12} is the relative deformation of the individual solitons' shapes due to their mutual interaction. For the sake of simplicity, we assume that $\Phi_1 = \Phi_2 = \Phi$, and we neglect the term Φ_{12} because, as we show, the nonlinear effects embodied by Φ_{12} can be accounted for by properly selecting the quasi-particle trajectories $X_1(t)$ and $X_2(t)$.

- Conjecture: By choosing appropriate paths $X_i(t)$ for the "centers" of the solitons (defined to be the location of the maximum of u_x for our purposes), the linear superposition of two single (undisturbed) solitons should be much closer to the exact two-soliton solution than the mere superposition of two single solitons following the linear trajectories $X_i(t) = c_i t$.
- To this end, we assume the solution looks like the linear superposition (5), rather than (3), keeping the trajectories of the centers of the solitons (*i.e.*, the location of the quasi-particles) **unknown**. Then, the **coarse-grain description** of the wave profile is just

$$u^{\pm}(x,t) = \Phi[x - X_1(t)] \pm \Phi[x - X_2(t)], \qquad \Phi(\eta) = 4\arctan[\exp(\eta)]. \tag{7}$$

- The "pseudo": The **pseudomass** of each solitary wave (*i.e.*, the mass of the corresponding quasi-particle) is defined as

$$\mathbb{M}_{ii} \overset{\text{def}}{=} \int_{-\infty}^{+\infty} \left[\Phi'(\xi_i)\right]^2 \mathrm{d}x = 4\int_{-\infty}^{+\infty} \operatorname{sech}^2(\xi_i)\,\mathrm{d}\xi_i = 8, \quad \xi_i = x - X_i(t), \quad i \in \{1,2\}, \tag{8}$$

and, similarly, the **crossmass** that the quasi-particles impart on each other is

$$\mathbb{M}_{12}(X_2 - X_1) \overset{\text{def}}{=} \pm\int_{-\infty}^{+\infty} \Phi'(x - X_1)\Phi'(x - X_2)\,\mathrm{d}x = \frac{\pm 8(X_2 - X_1)}{\sinh(X_2 - X_1)} \equiv \mathbb{M}_{21}(X_2 - X_1). \tag{9}$$

Moreover, the continuous **wave momentum** P of a solitary wave and the corresponding discrete wave momentum $\mathbb{P}$ (*i.e.*, the one obtained upon substituting the coarse-grain description for u into the definition of P) are

$$P \overset{\text{def}}{=} -\int_{-\infty}^{+\infty} u_x u_t\,\mathrm{d}x \quad \Longrightarrow \quad \mathbb{P} = (\mathbb{M}_{11} + \mathbb{M}_{12})\dot{X}_1 + (\mathbb{M}_{12} + \mathbb{M}_{22})\dot{X}_2. \tag{10}$$

Similarly, introducing $z(t) := X_2(t) - X_1(t)$, we obtain the **discrete** (or coarse-grain) Lagrangian of the system:

$$\mathbb{L} = \tfrac{1}{2}\mathbb{M}_{11}\left(\dot{X}_1^2 - 1\right) + \mathbb{M}_{12}(z)\left(\dot{X}_1\dot{X}_2 - 1\right) + \tfrac{1}{2}\mathbb{M}_{22}\left(\dot{X}_2^2 - 1\right) - V(z), \tag{11}$$

where V is the the potential energy term in Lagrangian, *i.e.*

$$V(z) := \int_{-\infty}^{+\infty} 1 - \cos\big(u^{\pm}\big)\,\mathrm{d}x. \tag{12}$$

- Equations of motion (EoM): The Euler–Lagrange equations for the minimization of functional (11) give the governing equations of the quasi-particles:

$$\begin{cases} \mathbb{M}_{11}\ddot{X}_1 + \mathbb{M}_{12}(z)\ddot{X}_2 = \mathbb{M}_{12}'(z) + V'(z), \\ \mathbb{M}_{12}(z)\ddot{X}_1 + \mathbb{M}_{22}\ddot{X}_2 = -\mathbb{M}_{12}'(z) - V'(z). \end{cases} \tag{13}$$

- N.B. The dynamics of each quasi-particle are affected by the presence of another, accelerating, quasi-particle near it. So, the dynamics are not strictly Newtonian, except in the trivial case of a single particle; a more appropriate term for the physics the above equations describe is **Machean**. (See [Ric83] for similar results in the presence of a driving force.)

Numerical Results and Conclusions

- The EoM can be recast into a ODE for $z(t)$ [$\equiv X_2(t) - X_1(t)$], which we solve numerically using Mathematica's `DESolve`. Then, each quasi-particle's trajectory can be recovered.

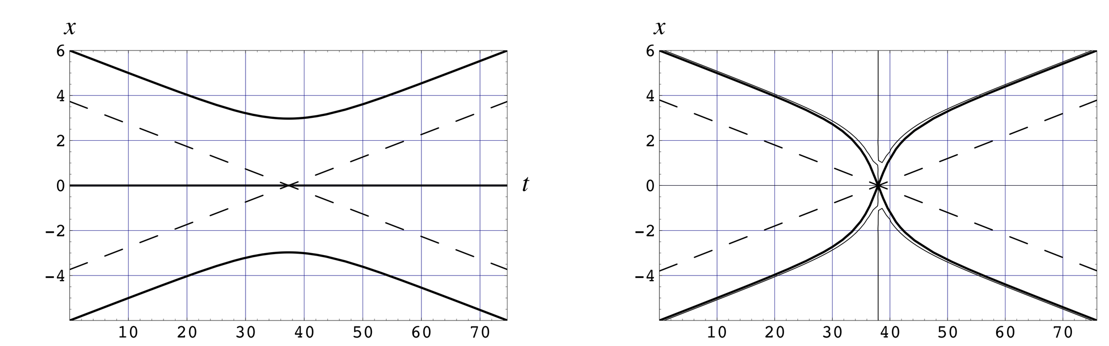

Figure 2: Thick solid lines represent the trajectories of the interacting quasi-particles for the soliton-soliton (left) and soliton-antisoliton (right) case. Dashed lines represent the trajectories of the noninteracting quasi-particles. Thin solid lines represent the the zero contour of the two-soliton solution (*i.e.*, the location of the centers of the soliton).

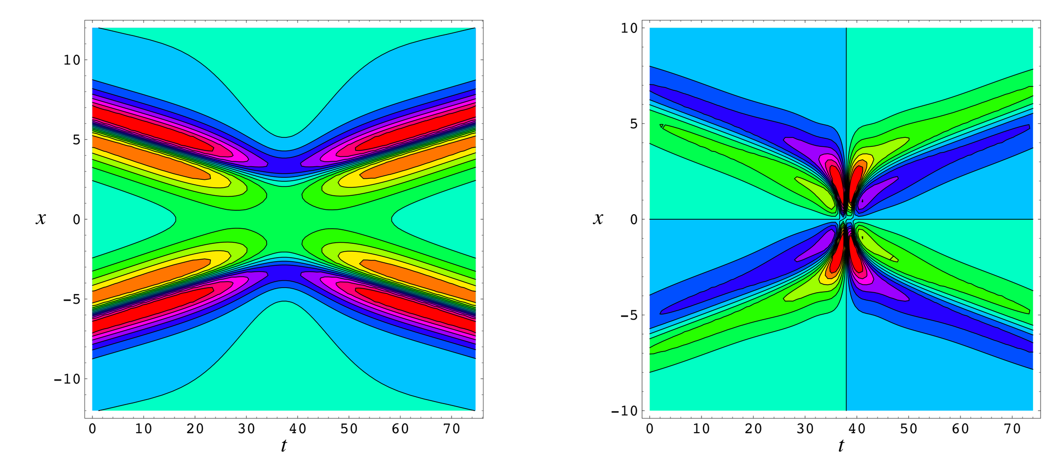

Figure 3: Error in the coarse-grain description of interacting solitons. (Left) S-S case, 13 contours from -0.02 to 0.02. (Right) S-A case, 13 contours from -0.3 to 0.3.

- A consistent way to treat solitons as quasi-particles is proposed. In particular:
  - The passage from the continuous to a discrete description of the dynamics of coherent structures in nonlinear wave equations has been demonstrated via the **coarse-grain** simplification.
  - The notion of **pseudomass** of a coherent structure is introduced and assigned as the mass of a quasi-particle located at the coherent structure's "center." The pseudo-Newtonian law governing the dynamics of the centers (*i.e.*, of the quasi-particles) is derived, and its predictions agree with the **exact** dynamics [BS77].
  - The proposed approach can serve as a basis of the (approximate) study of coherent structures (two or more) in **non-integrable** nonlinear wave equations.

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